Harm Anal.

cl 2/1

We've proved that if $f \in L^1(\mathbb{R}^n)$ and $\operatorname{supp} f := \overline{\{x;\, f(x) \neq 0\}} \subseteq B_R$

then $\hat{f} \in C^\infty$.

Prop 1.4 $f \in C^N$ and $D^\alpha f \in L^1$ $\forall\ 0 \le |\alpha| \le N$.

Then $\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$, when $|\alpha| \le N$

& $|\hat{f}(\xi)| \le C(1+|\xi|)^{-N}$ for some $C>0$.

Prep. cut-off functions

- $g(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x>0 \\ 0, & \text{if } x \le 0 \end{cases}$, then $g(x) \in C^\infty(\mathbb{R})$

Pf: $g^{(k)}(x) = \begin{cases} P_k\left(\frac{1}{x}\right) e^{-\frac{1}{x}} \\ 0, & x \le 0 \end{cases}$ $\forall k \ge 0$, $P_k(t)$ pol.

$\Rightarrow g^{(k)} \in C(\mathbb{R})$ con't

- $f(x) = g(1-|x|^2) \in C^\infty(\mathbb{R}^n)$: by Chain rule

and $f(x) = 0$ if $|x| \ge 1$, $f(x) > 0$ if $|x| < 1$

$\Rightarrow \operatorname{supp} f \subseteq B(0,1)$, $f \in C_0^\infty(\mathbb{R}^n)$

Let $\varepsilon > 0$, $f_\varepsilon(x) = \varepsilon^{-n} f\left(\frac{x}{\varepsilon}\right)$, then $\operatorname{supp} f_\varepsilon \subseteq B(0,\varepsilon)$

& $\int_{\mathbb{R}^n} f_\varepsilon = \int f < \infty$, $\forall \varepsilon > 0$

Convolution Let $f \in L^1$, $g \in L^\infty$, define $f * g(x) = \int \underbrace{f(x-y) g(y)}_{\in L^1} dy$

Prop: If $f \in L^1$, $g \in C_0^\infty$ then $f * g \in C^\infty$

Prop: $f * g = g * f$, $(f*g)*h = f*(g*h)$

$$f*g(x) = \int f(\underbrace{x-y}_{z}) g(\underset{x-z}{y}) dy = \int f(z)\, g(x-z)\, dz = g*f(x)$$

Pf

$$\partial_{x_j}(f*g)(x) = \partial_{x_j} \int f(y)\, g(x-y)\, dy =$$

$$= \lim_{h \to 0} \int f(y) \underbrace{\frac{g(x-y+he_j) - g(x-y)}{h}}_{\Delta_{he_j} g(x-y)} dy$$

, where $\Delta_{he_j} g(x) = \dfrac{g(x+he_j) - g(x)}{h}$ $\quad g_j(h) - g_j(0)$

We need an estimate for $\Delta_h^j g(x) = \frac{1}{h} \int_0^h \partial_{x_j} g(x + t e_j)\, dt$, $t = h\tau$

$$\int_0^1 \partial_{x_j} g(x + \tau h e_j)\, d\tau$$

$x \quad x+he_j$

$\le \|\partial_{x_j} g\|_\infty \le C$

Then by LDC, as $\lim_{h \to 0} \Delta_h^j g(x) = \partial_{x_j} g(x)$

$$= \int f(y)\, \partial_{x_j} g(x-y)\, dy = f * \partial_{x_j} g$$

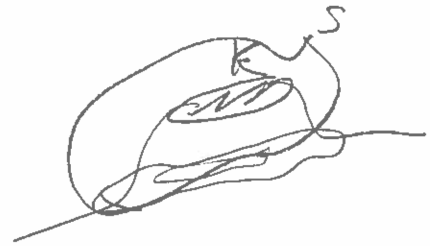
Thus $\forall \alpha$: $\exists\, D^\alpha(f*g) = f*(D^\alpha g) \Longrightarrow f*g \in C^\infty(\mathbb{R}^n)$

$\square$

Prop 1

Let $K \subseteq \mathbb{R}^n$ compact, $U \subseteq \mathbb{R}^n$ open st $K \subseteq U$.

Then $\exists\ \phi \in C_0^\infty(\mathbb{R}^n)$ st. $\phi(x)=1\ \forall x\in K$ and $\phi(x)=0\ \forall x \notin S$



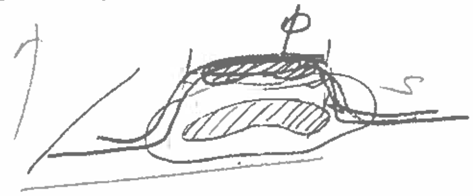
Pf Let $S = \mathbb{R}^n \setminus U$ closed, $K\cap S=\emptyset$

$\Rightarrow \operatorname{dist}(K,S) = \inf\{|x-y|, \dots\} = d>0$.

Let $\varepsilon < \frac{d}{2}$, then $K_{2\varepsilon} = \{x;\ \exists y\in K:\ |x-y|\le 2\varepsilon\} \subseteq U$.

Let $\phi_\varepsilon \in C_0^\infty$ st. $\phi_\varepsilon(0)>0$, $\operatorname{supp}\phi_\varepsilon \subseteq B(0,\varepsilon)$

and $\int \phi_\varepsilon(x)\,dx = 1$.



Let $g = 1_{K_\varepsilon}$ the Char. fn of K_ε, and let $f := g * \phi_\varepsilon \in C_0^\infty$.

If $x\in K$ then $x-y\in K_\varepsilon\ \forall y\in B_\varepsilon \Rightarrow g(x-y)=1$

$$f_\varepsilon(x) = \int g(x-y)\,\phi_\varepsilon(y)\,dy = \int_{B(0,\varepsilon)} g(x-y)\,\phi_\varepsilon(y) = \int \phi_\varepsilon(y) = 1$$

If $x \notin S$ then $x-y\notin K_\varepsilon\ \forall y\in B_\varepsilon \Rightarrow g(x-y)=0$

$$\Rightarrow f(x) = \int_{B_\varepsilon} g(x-y)\,\phi_\varepsilon(y)\,dy = 0$$

$\square$

Note For any func $h(x)$ we have $h\cdot f_\varepsilon = h$ on K

$h\cdot f_\varepsilon = 0$ outside S.

Harm. Anal

<u>Lemma</u> If $f \in C^N(\mathbb{R}^n)$, $D^\alpha f \in L^1$ $\forall\, |\alpha| \le N$, Then

$$\| D^\alpha(\phi_k f) - D^\alpha f \|_1 \to 0 \quad \text{as } k \to \infty, \text{ where } \phi_k(x) = \phi\left(\tfrac{x}{k}\right)$$

<u>Pf:</u> • $\| \phi_k D^\alpha f - D^\alpha f \|_1 \to 0$ as $k \to \infty$ ($\phi \equiv 1$ on $|x| \le 1$, $\phi \equiv 0$ on $|x| \ge 2$
by LDC $0 \le \phi \le 1$)

• Leibnitz (gen prod) rule:

$$D^\alpha(\phi_k f) - \phi_k D^\alpha f = \sum_{0 < \beta \le \alpha} c_\beta (D^{\alpha-\beta} f) \cdot (D^\beta \phi_k)$$

$$\Rightarrow \| D^\alpha(\phi_k f) - \phi_k D^\alpha f \|_1 \le C \sum_{0 < \beta \le \alpha} \| D^{\alpha-\beta} f \|_1 \, \| D^\beta \phi_k \|_\infty$$

as $D^\beta \phi\left(\frac{x}{k}\right) = k^{-|\beta|} (D^\beta \phi)\left(\frac{x}{k}\right)$. $\le \frac{C}{k} \sum_{0 < \beta \le \alpha} \| D^{\alpha-\beta} f \|_1 \to 0$ as $k \to \infty$

• $\| D^\alpha(\phi_k f) - D^\alpha f \|_1 \le \| D^\alpha(\phi_k f) - \phi_k (D^\alpha f) \|_1 + \| \phi_k D^\alpha f - D^\alpha f \|_1$

<u>Pf of Prop:</u> (i.e. $\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$) $\square$

• $D^\alpha(\phi_k f) \xrightarrow{L^1} D^\alpha f \Rightarrow \widehat{D^\alpha(\phi_k f)}(\xi) \to \widehat{D^\alpha f}(\xi)$

$\widehat{\phi_k f}(\xi) \to \hat{f}(\xi)$

• WLOG can assume f is comp. supported

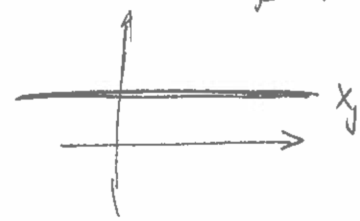
• Let $\alpha = (0, \dots, 1, \dots, 0)$ i.e. $D^\alpha f = \partial_{x_j} f$ then use induction

Since f has compact support:

$$\int \partial_{x_j} f(x)\, e^{-2\pi i x\cdot\xi}\, dx = -\int f(x)\, \partial_{x_j}\left(e^{-2\pi i x\cdot\xi}\right) dx$$

$$= (2\pi i \xi_j) \int f(x) e^{-2\pi i x\cdot \xi}\, dx = (2\pi i \xi_j)\, \hat{f}(\xi)$$

(follows from 1-dim int. by parts)



- Induction on $|\alpha| = \alpha_1 + \dots + \alpha_n$

$\square$

Cor. If $p(\xi) = p(\xi_1, \dots \xi_n)$ is a pol., then for $p(D) = \sum_{|\alpha| \le N} a_\alpha D^\alpha$

diff. op $\widehat{p(D) f}(\xi) = p(2\pi i \xi)\, \hat{f}(\xi)$

- This gives rise to multiplier op's if $a(\xi) \in C^\infty \cap L^\infty$

then $\widehat{a(D) f}\,(\xi) := a(2\pi i \xi)\, \hat{f}(\xi) \leftarrow$ first Four. inversion

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Schwarz - space