Let E be an ellipse and let its dual ellipse be:

$E^* = E_k(r_1, \dots, r_n)$.

Define the "smoothed" ~~normalized~~ indicator function of E^*:

$$\varphi_{E^*}(x) := \left(1 + \sum_{j=1}^{n} \frac{|(x-k)\cdot e_j|^2}{r_j^2}\right)^{-N}, \text{ for some fixed } N \geq 1$$

<u>Prop</u> Let $f \in L^1 + L^2$, $\operatorname{supp} \hat{f} \subseteq E$. Then $\forall y \in E^*$,

$$|f(y)| \leq C_N \frac{1}{|E^*|} \int |f(x)| \, \varphi_{E^*}(x) \, dx$$

<u>Pf</u>

(i) Assume $E = E^* = B_1$. Choose $\psi \in S$ s.t. $f = f * \psi$.

Then $$|f(y)| \leq \int |f(x)\psi(y-x)| dx \leq C_N \int |f(x)| (1 + |y-x|^2)^{-N} dx.$$

If $y \in B_1$ so $|y| \leq 1$ then $1 + |y-x|^2 \geq \frac{1}{5}(1 + |x|^2)$ for all x (take $|x| \leq 2$ and $|x| \geq 2$ separately), thus

$$|f(y)| \leq C_N' \int |f(x)| (1 + |x|^2)^{-N} = C_N' \int |f(x)| \, \varphi_{E^*}(x) \, dx$$

(ii) $E = E_0(r_1^{-1}, \dots, r_n^{-1})$, $E^* = E_k^*(r_1, \dots, r_n)$. . .

Let $T : (x_1, \dots, x_n) \mapsto (r_1 x_1, \dots, r_n x_n)$, $g(x) := f(Tx + k)$

Then $|\hat{g}(\xi)| = (\det T)^{-1} |\hat{f}(T^{-1}\xi)| \neq 0$ when $T^{-1}\xi \in E$

but $T^{-1}\zeta \in E \Leftrightarrow \zeta \in T(E) = B_1$

(Indeed $\eta = (\eta_1, \dots, \eta_n) \in E \Leftrightarrow \sum_{j=1}^{n} (r_j \cdot \eta_j)^2 \leq 1 \Leftrightarrow T\eta = (r_1\eta_1, \dots, r_n\eta_n) \in B_1$.

Thus $\operatorname{supp} \hat{g} \subseteq B_1$ which implies $\forall z \in B_1$;

$$|f(Tz+k)| = |g(z)| \leq c_N \int |g(y)|\, \phi(y)\, dy = \int f(Ty+k)\, \phi(y)\, dy =$$

Let $x := Ty+k$, $y = T^{-1}(x-k)$

$$= c_N (\det T)^{-1} \int |f(x)|\, \phi\left(\frac{x_1-k_1}{r_1}, \dots, \frac{x_n-k_n}{r_n}\right) dx$$

$$= c_N \frac{1}{|E^*|} \int f(x)\, \phi_{E^*}(x)\, dx$$

But $y \in E^*$ then $y = Tz+k$ for $z = T^{-1}(y-k) \in B_1$

thus $|f(y)| \leq c_N \frac{1}{|E^*|} \int f(x)\, \phi_{E^*}(x)\, dx$

$\square$

Stationary phase

Let ϕ be smooth, and let $a \in C_0^\infty$ (amplitude), ϕ (phase).

For $\lambda > 1$ consider the oscillatory integral

$$I(\lambda) := \int_{\mathbb{R}^n} e^{-\pi i \lambda \phi(x)} a(x)\, dx \;;$$

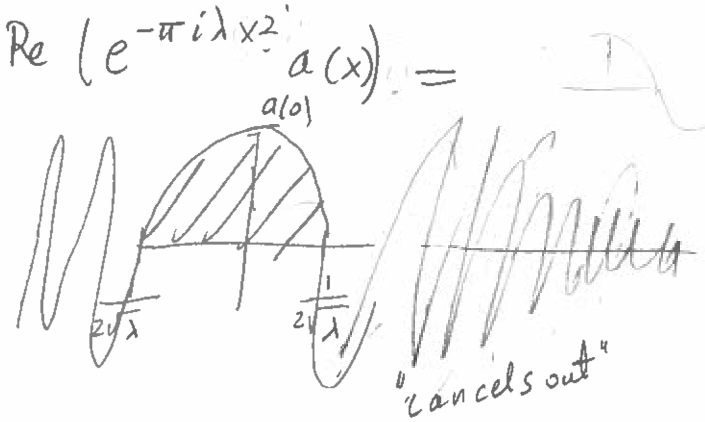
and would like to understand $I(\lambda)$ asymptotically as $\lambda \to +\infty$.

Note $(n=1)$

- $|I(\lambda)| \le \int |a(x)|dx \le C$ uniformly in λ
- $\phi(x) = x$, $n=1$ $\Rightarrow$ $I(\lambda) = \hat{a}(\lambda/2) \le C_N (1+\lambda)^{-N}$; $\forall N > 0$

  $\phi'(0) \ne 0$ $\Rightarrow$ $I(\lambda)$ rapidly decreasing.
- $\phi(x) = x^2$, $a(0) \ne 0$ then $\mathrm{Re}\left(e^{-\pi i \lambda x^2} a(x)\right) =$

  $= \cos(\pi \lambda x^2)\, a(x) =$



we expect $\Rightarrow$ $I(\lambda) \approx a_0 \lambda^{-\frac{1}{2}}$ as $\lambda \to \infty$

$\phi'(0) = 0$ but $\phi''(0) \ne 0$

The case $n=1$

Prop 12.1 (localization) Let $\phi, a \in C^\infty(\mathbb{R})$, $\operatorname{supp} a \subseteq (a,b)$.

If $\phi'(x) \neq 0$ for all $x \in [a,b]$, then $\forall N \in \mathbb{N}$, $\forall \lambda \geq 1$

$$|I(\lambda)| \leq C_N \lambda^{-N}$$

Note, we say $I(\lambda) = O(\lambda^{-N})$ "rapidly decreasing"

Proof $\frac{d}{dx}(e^{i\lambda\phi}) = (i\lambda\phi')\, e^{i\lambda\phi}$, thus if $D = \frac{1}{i\lambda\phi'} \cdot \frac{d}{dx}$

then $D(e^{i\lambda\phi}) = e^{i\lambda\phi} \Rightarrow D^N(e^{i\lambda\phi}) = e^{i\lambda\phi}$

Now, writing $\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx$ we have with $\operatorname{supp} g \subseteq (a,b)$

$$\langle Df, g\rangle = \int_a^b Df(x) \cdot g(x)\, dx = \int_a^b \frac{d}{dx} f(x)\, (i\lambda\phi')^{-1} g(x)\, dx$$

$$\int_a^b f(x)\, \frac{i}{\lambda} \frac{d}{dx}\left(\frac{g(x)}{\phi'(x)}\right) dx = \langle f, D^t g\rangle$$

with $D^t g = \frac{i}{\lambda} \frac{d}{dx}\left(\frac{g}{\phi'}\right) = \frac{i}{\lambda} \Delta g$

This implies that $\forall N \in \mathbb{N}$, $\langle D^N f, g\rangle = \langle f, (D^t)^N g\rangle$,

thus

$$I(\lambda) = \langle e^{i\lambda\phi}, a\rangle = \langle D^N(e^{i\lambda\phi}), a\rangle =$$

$$= \langle e^{i\lambda\phi}, (D^t)^N a\rangle = \left(\frac{i}{\lambda}\right)^N \langle e^{i\lambda\phi}, \Delta^N a\rangle$$

$$\Rightarrow |I(\lambda)| \leq \lambda^{-N} \|\Delta^N a\|_1 \leq C_N \lambda^{-N} \qquad \square$$

<u>Proof 2</u> If $\phi'(x) \neq 0 \ \forall\, a \leq x \leq b$ then either $\phi'(x) > 0 \ \forall\, a \leq x \leq b$

or $\phi'(x) < 0 \ \forall\, a \leq x \leq b$.

Assume $\phi'(x) > 0$ (otherwise replace $\phi(x)$ with $-\phi(x)$).

Then $\phi : [a,b] \twoheadrightarrow [c,d]$ is mon increasing, onto

so by doing a change of variables $y = \phi(x)$ for $x \in (a,b)$
$\exists\, x = \phi^{-1}(y)$ for $y \in (c,d)$

$$I(\lambda) = \int_a^b e^{\pi i \lambda \phi(x)} a(x)\,dx = \int_c^d e^{i\lambda y} (a \circ \phi^{-1}(y)) \frac{1}{\phi'(\phi^{-1}(y))}\,dy$$

$$= \int_c^d e^{\pi i \lambda y}\, b(y)\,dy, \quad \text{with} \quad b(y) = \frac{a(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))},$$

as $y = \phi(x) \implies dy = \phi'(x)\,dx \implies dx = \frac{1}{\phi'(x)}\,dy = \frac{1}{\phi'(\phi^{-1}(y))}\,dy$.

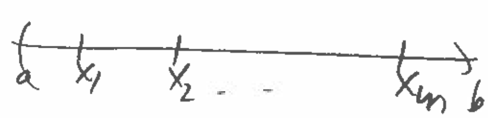
But $b \in C_0^\infty(\mathbb{R})$, hence

$$I(\lambda) = \hat{b}\left(-\frac{\lambda}{\pi}\right) = O_N(\lambda^{-N}) \quad \forall N.$$

<u>Note</u>

- If $\phi(x)$ has finitely many critical points $a < x_1 < x_2 < \dots < x_m < b$, where $\phi'(x_i) = 0$, then these critical points determine the behavior of $I(\lambda)$, as $\lambda \to +\infty$.

- Indeed, there is a partition of unity $\psi_1, \dots, \psi_m \in C_0^\infty$ such that

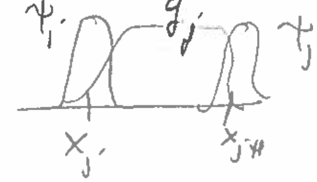


- $(\psi_1 + \dots + \psi_m)(x) = 1 \quad \forall \quad x \in (a,b)$
- $\psi_j(x_j) = 1$, $\operatorname{supp} \psi_j \subseteq (x_{j-1}, x_{j+1})$

Writing $a_j(x) = \psi_j(x) a(x)$ we have $a = \sum_{j=1}^m a_j$, hence

$$I(\lambda) = \int e^{i\lambda\phi(x)} \sum_{j=1}^m a_j(x)\,dx = \sum_{j=1}^m \int e^{i\lambda\phi(x)} a_j(x)\,dx = \sum_{j=1}^m I_j(\lambda).$$

- We can make $\operatorname{supp} \psi_j \subseteq (x_j - \varepsilon, x_j + \varepsilon)$ $\forall j \le m$, by adding functions g_j, but the contribution of the integrals



$$\tilde{I}_j(\lambda) := \int e^{i\lambda\phi(x)} (g_j a)(x)\,dx = O_N(\lambda^{-N})$$

thus $I(\lambda) = \sum_{j=1}^m I_j(\lambda) + O(\lambda^{-N}) \Rightarrow$ localised to critical points, where $\phi'(x) = 0$.

2. Non-deg. critical points. Suppose $\phi'(0) = 0$, $\phi''(0) \neq 0$.

Let $a \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} a \subseteq (-\varepsilon, \varepsilon)$ suff. small.

Then $$I(\lambda) := \int_{\mathbb{R}} e^{\pi i\lambda\phi(x)} a(x)\,dx$$

<u>Prop 12.2.</u> Let $I(\lambda) := \int_{\mathbb{R}} e^{-\pi i \lambda x^2} a(x)\,dx$. Then $\forall N \in \mathbb{N}$

$$I(\lambda) = e^{\frac{-\pi i}{4}} \lambda^{-\frac{1}{2}} \left(a(0) + \sum_{j=1}^N \lambda^{-j} a_j + O(\lambda^{-N-1}) \right)$$

<u>Note</u> This gives a full asymptotic.

Pf Let $g_\lambda(x) = e^{-\pi i \lambda x^2}$ (imaginary Gaussian);

we proved $\hat{g}_{i\lambda}(\xi) = \lambda^{-\frac{1}{2}} e^{-\frac{\pi i}{4}} e^{\frac{\pi i \xi^2}{\lambda}}$ in the distribution sense, i.e.

$$I(\lambda) = \int g_{i\lambda}(x)\, a(x)\, dx = \int \hat{g}_{i\lambda}(\xi)\, \hat{a}(\xi)\, d\xi =$$

$$= e^{-\frac{\pi i}{4}} \lambda^{-\frac{1}{2}} \int e^{\frac{\pi i \xi^2}{\lambda}}\, \hat{a}(\xi)\, d\xi$$

Now write $e^{\frac{\pi i \xi^2}{\lambda}} = 1 + \sum_{j=1}^{\infty} \frac{(\pi i)^j \xi^{2j}}{j!\, \lambda^j}$, then

$$I(\lambda) = e^{-\frac{\pi i}{4}} \lambda^{-\frac{1}{2}} \left(a_0 + \sum_{j=1}^{\infty} \lambda^{-j} \left(\int \frac{(\pi i)^j}{j!} \xi^{2j} \hat{a}(\xi)\, d\xi \right) \right)$$

$$= e^{-\frac{\pi i}{4}} \lambda^{-\frac{1}{2}} \left(a_0 + \sum_{j=1}^{\infty} a_j \lambda^{-j} \right)$$

Note that $|a_j| \leq \frac{K^{2j} \pi^j}{j!} \|\hat{a}\|_1$, if $\operatorname{supp} a \subseteq (-K, K)$

as $|\xi^{2j} \hat{a}(\xi)| \leq K^{2j} |\hat{a}(\xi)|$ on $\operatorname{supp} a \subseteq (-K, K)$,

thus the series converges. $\square$

Note • a_j can be explicitely computed as

$$\widehat{D^j a}(\xi) = (2\pi i \xi)^j \hat{a}(\xi)$$

• If $\phi(0) = \phi'(0) = 0$ but $\phi''(0) \neq 0$, then with $c = \frac{\phi''(0)}{2}$

$$\phi(x) = c \cdot x^2 (1 + \varepsilon(x)) \quad \text{with } \varepsilon(x) \to 0 \text{ as } x \to 0$$

thus $y = x\, (1 + \varepsilon(x))^{\frac{1}{2}}$ is a diffeom near 0.