Uncertainty Principle

If $\hat{f} \in L^p(\mathbb{R}^n)$, supp $\hat{f} \subseteq B_R$, then

(1) $\hat{f} \in L^1(\mathbb{R}^n)$, $f(x) = \int e^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi$

(2) Let $0<\varepsilon<1$, If $|x-y| \le \varepsilon R^{-1}$ then $|f(x)-f(y)| \le C\varepsilon$

"f is approx constant at scale εR"

Pf

(1) $$\int_{B_R} |\hat{f}| \le \int_{B_R} |\hat{f}|\, \mathbb{1}_{\{|\hat{f}|\le 1\}} + \int_{B_R} |\hat{f}|\, \mathbb{1}_{|\hat{f}|\ge 1}$$

$$\le |B_R| + \int |\hat{f}|^p \le |B_R| + \|f\|_p^p < \infty$$

(2) Let $\psi_k = \hat{\Phi}_k \in \mathcal{S}(\mathbb{R}^n)$ s.t. $\psi_k \xrightarrow{L^1} \hat{f}$, $\psi_k \xrightarrow{L^2} \hat{f}$

This implies $\hat{\Phi}_k \xrightarrow{L^2} \hat{f} \Rightarrow \Phi_k \xrightarrow{L^2} f$

$$\Phi_k(x) = \int e^{2\pi i x\cdot\xi} \psi_k(\xi)\, d\xi \xrightarrow{L^\infty} \int e^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi = \tilde{f}$$

Choosing subseq $\Rightarrow$ $\tilde{f}(x) = f(x)$ for a.e. x

(2) $$|f(x)-f(y)| \le \int |e^{2\pi i x\cdot\xi} - e^{2\pi i y\cdot\xi}|\, |\hat{f}(\xi)|\, d\xi$$

$$\le \int \underbrace{2\pi|x-y|\,|\xi|}\, |\hat{f}(\xi)|\, d\xi \le C\varepsilon \|\hat{f}\|_1 \le C_1 \varepsilon$$

Uncertainty Principle

If f is supp on B_R, then $\hat{f}$ (can be regarded) "is constant" on a ~~[illegible]~~ scale R^{-1}. Note $f_R = f(\frac{x}{R}) \Rightarrow \hat{f_R} = R\cdot\hat{f}(R\cdot\xi) \to$

Prop 5.1 (L^2 Bernstein's ineq.) If $f \in L^2$ and $\operatorname{supp}\hat{f} \subseteq B_R$, then $f \in C^\infty$ and $\|D^\alpha f\|_2 \leq (2\pi R)^{|\alpha|} \|f\|_2$

Pf: $\hat{f} \in L^1$ and $f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi}$ $(*)$

(as choose $\hat{\Phi}_k \in S$ s.t. $\hat{\Phi}_k \xrightarrow{L^1} \hat{f}$, $\hat{\Phi}_k \xrightarrow{L^2} \hat{f}$)

$\Longrightarrow$ $\Phi_k \xrightarrow{L^2} f$ and $\hat{\Phi}_k \xrightarrow{L^\infty} \int \hat{f}(\xi)e^{2\pi i x\xi} d\xi$

Then we also have $\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$

$\Longrightarrow \|D^\alpha f\|_2 = \|\widehat{D^\alpha f}\|_2 = \|(2\pi\xi)^\alpha \hat{f}\|_2 \leq (2\pi R)^{|\alpha|} \|f\|_2$

□

Lemma 5.2. $\exists\, \Phi \in S$ s.t. $\forall f \in L^1 + L^2$ with $\operatorname{supp}\hat{f} \subseteq B_R$ one has $f = \Phi_{1/R} * f$.

Pf: Let $\phi \in S$ st. $\hat{\phi} \equiv 1$ on B_1. Then $\hat{\Phi}_{1/R}(\xi) = \hat{\Phi}(\xi/R)$

$\hat{f} = \hat{f}\cdot\hat{\Phi}_{1/R} \Longrightarrow f = f * \phi_{1/R}$

□

Prop. 5.3 (Bernstein ineq. disc.) $f \in L^1 + L^2$, $\hat{f}$ supp. B_R

Then

(i) $\forall \alpha$, $1 \le p \le \infty$ $\quad \|D^\alpha f\|_p \le (CR)^{|\alpha|} \|f\|_p$

(ii) $1 \le p \le q \le \infty$ $\quad \|f\|_q \le C R^{n(\frac{1}{p}-\frac{1}{q})} \|f\|_p$.

Pf: Let $\psi = \Phi_{1/R}$, then • $\|\psi\|_r = C R^{n/r'}$ $\quad \frac{1}{r} + \frac{1}{r'} = 1$

• $\|\nabla \psi\|_1 = R \|\Phi\|_1$

(i) $$D_j(\psi * f) = D_j \psi * f$$

$$\Rightarrow \|D_j f\|_p = \|D_j(\psi * f)\|_p \le \|D_j \psi\|_1 \|f\|_p \le CR \|f\|_p$$

+ induction

(ii) Young's ineq (interpolation) $\quad \frac{1}{q} = \frac{1}{p} - \frac{1}{r'}$

$$\|f\|_q = \|\psi * f\|_q \le \|\psi\|_r \|f\|_p \lesssim R^{\frac{n}{r'}} \|f\|_p$$

$$= R^{n(\frac{1}{q} - \frac{1}{p})} \|f\|_p$$

If $\hat{f}$ supported on B_R, the ball of radius R, centered at 0 then f is approximately constant at scales $\ll R^{-1}$.

Lemma $\operatorname{supp}\hat{f} \subseteq B_R$, $\varepsilon > 0$. If $|x-y| \le \varepsilon R^{-1}$ then $|f(x)-f(y)| \le C\varepsilon$.

Pf $f(x) = \int_{B_R} e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,d\xi$;

$$|x-y| \le \varepsilon R \Rightarrow |e^{2\pi i x\cdot\xi} - e^{2\pi i y\cdot\xi}| = |e^{2\pi i (x-y)\cdot\xi} - 1| \le$$

$$\le 2\pi |x-y|\cdot|\xi| \le 2\pi\varepsilon R^{-1}\cdot R = 2\pi\varepsilon \quad \text{for } |\xi| \le R$$

$$\Rightarrow |f(x)-f(y)| \le \int_{B_R} |e^{2\pi i x\cdot\xi} - e^{2\pi i y\cdot\xi}|\,|\hat{f}(\xi)|\,d\xi \le \varepsilon\, 2\pi \|\hat{f}\|_1 = C\varepsilon$$

$\square$

Prop 11.1 (L^2-Bernstein's ineq.)

Assume $f \in L^2$, $\operatorname{supp}\hat{f} \subseteq B_R$.

Then $f \in C^\infty$, and $\|D^\alpha f\|_2 \le (2\pi R)^{|\alpha|}\|f\|_2$

Pf: $f \in L^2 \Rightarrow \hat{f} \in L^2$, $\operatorname{supp}\hat{f} \subseteq B_R \Rightarrow \hat{f} \in L^1 \overset{(why?)}{\Longrightarrow} f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi$

$$\Rightarrow D^\alpha f(x) = (2\pi i)^{|\alpha|}\int \xi^\alpha \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi$$

$$\Rightarrow \|D^\alpha f\|_2 = (2\pi)^{|\alpha|}\|\xi^\alpha \hat{f}\|_2 \le (2\pi R)^{|\alpha|}\|\hat{f}\|_2 = (2\pi R)^{|\alpha|}\|f\|_2 \quad \square$$

Hölder - for 3-functions Let $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. then $\forall f,g,h$

$$\left|\int_\Omega fgh\,d\mu\right| \le \left(\int|f|^p\right)^{\frac{1}{p}}\left(\int|g|^q\right)^{\frac{1}{q}}\left(\int|h|^r\right)^{\frac{1}{r}} = \|f\|_p\|g\|_q\|h\|_r$$

Pf: $\frac{1}{p}+\frac{1}{q}=1-\frac{1}{r}=\frac{1}{r'} \Rightarrow \frac{r'}{p}+\frac{r'}{q}=1$, $p_1:=\frac{p}{r'}$, $q_1:=\frac{q}{r'}$

$$\frac{1}{p_1}+\frac{1}{q_1}=1$$

$$\left|\int_\Omega (fg)\cdot h\right| \le \|fg\|_{r'}\|h\|_r = \left(\int |f|^{r'}|g|^{r'}\right)^{\frac{1}{r'}}\|h\|_r$$

$$F:=|f|^{r'};\ G:=|g|^{r'}$$

$$\int FG \le \left(\int|F|^{p_1}\right)^{\frac{1}{p_1}}\left(\int|G|^{q_1}\right)^{\frac{1}{q_1}} =$$

$$=\left(\int|f|^p\right)^{\frac{r'}{p}}\left(\int|g|^q\right)^{\frac{r'}{q}}$$

$$\Rightarrow \left|\int fgh\right| \le \|f\|_p\|g\|_q\|h\|_r$$

Thm (Young's ineq.) Let $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$ and assume

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1 \quad \text{then}$$

$$\|f*g\|_r \le \|f\|_p\|g\|_q$$

Pf: Since $\|f*g\|_r = \sup_{\|h\|_{r'}\le 1}\left|\int (f*g)h\right|$ enough to show that

$$\left|\int(f*g)\cdot h\right| = \left|\iint f(x)\,g(y-x)\,h(y)\right| \le \|f\|_p\|g\|_q\|h\|_{r'}$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r'}=\frac{1}{p}+\frac{1}{q}+1-\frac{1}{r}=2$.

11/3

Write r for r', so $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$.

Then $p\left(2-\frac{1}{q}-\frac{1}{r}\right)=q\left(2-\frac{1}{p}-\frac{1}{r}\right)=r\left(2-\frac{1}{p}-\frac{1}{q}\right)=1$, thus by Hölder

$$\iint f(x)\, g(y-x)\, h(y)\, dx\, dy = \iint$$

$$= \iint \left(f(x)^p\, g(y-x)^q\right)^{1-\frac{1}{r}=\frac{1}{p_1}} \cdot \left(f(x)^p\, h(y)^r\right)^{1-\frac{1}{q}=\frac{1}{q_1}} \left(g(y-x)^q\, h(y)^r\right)^{1-\frac{1}{p}=\frac{1}{r_1}} dx\, dy$$

$$\le \Big(\underbrace{\iint f(x)^p\, g(y-x)^q}_{F}\Big)^{1-\frac{1}{r}} \Big(\underbrace{\iint f(x)^p\, h(y)^r}_{G}\Big)^{1-\frac{1}{q}} \Big(\underbrace{\iint g(y-x)^q\, h(y)^r}_{H}\Big)^{1-\frac{1}{p}}$$

$$= \|f\|_p^{p(1-\frac{1}{r})}\, \|g\|_q^{q(1-\frac{1}{r})}\, \|f\|_p^{p(1-\frac{1}{q})}\, \|h\|_r^{r(1-\frac{1}{q})}\, \|g\|_q^{q(1-\frac{1}{p})}\, \|h\|_r^{r(1-\frac{1}{p})}$$

$$= \|f\|_p\, \|g\|_q\, \|h\|_r$$

$\square$

Lemma Let $f \in L^1 + L^2$ and assume that $\operatorname{supp} \hat{f} \subseteq B_R$.

Then $\exists\, \phi \in S(\mathbb{R}^n)$ s.t. $f = f * \phi_{R^{-1}}$, where $\phi_{R^{-1}} = R^n \phi(xR)$

Pf $\widehat{\phi_{R^{-1}}}(\xi) = \hat{\phi}(R^{-1}\xi)$ (by ch. of var $x := Rx$)

Choose ϕ s.t. $\hat{\phi} \equiv 1$ on $B_1 = \{|x| \le 1\} \Rightarrow \hat{\phi}(R^{-1}\xi) \equiv 1$ on B_R

$\Rightarrow \hat{f} = \hat{f}\, \widehat{\phi_{R^{-1}}} \Rightarrow f = f * \phi_{R^{-1}}$

Prop 11.3 (Bernstein ineq. for the disc.)

Let $f \in L^1 + L^2$, $\operatorname{supp} \hat{f} \subseteq B_R$. Then

(i) $\forall \alpha$ $\forall$ $1 \le p \le \infty$: $\|D^\alpha f\|_p \le C\, R^{|\alpha|} \|f\|_p$

(ii) $\forall$ $1 \le p \le q \le \infty$ $\|f\|_q \le C\, R^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_p$

Pf • Let $\psi = \phi_{R^{-1}}$; then $\psi(x) = R^n \psi(Rx)$

$\Rightarrow$ $\|\psi\|_r = \left(\int |\psi(x)|^r\right)^{\frac{1}{r}} = R^n \left(\int |\phi(Rx)|^r dx\right)^{\frac{1}{r}} \underset{y:Rx}{=} R^{n(1-\frac{1}{r})} \|\phi\|_r =$

$= C R^{n/r'}$

• $\nabla \psi := \sum_{j=1}^{1} D_j \psi = R^{n+1} \sum_{j=1}^{n} D_j \phi(Rx)$

$\phi(Rx)$

$\Rightarrow$ $\|D_j \psi\|_1 = R^{n+1} \int \left| \sum_{j=1}^{n} (D_j \phi) \underbrace{(Rx)}_{\psi} \right| dx = R \|D_j \phi\|_1$

$dx = R^{-n} dy$

• $\overset{\text{induct.}}{\Longrightarrow}$ $\|D^\alpha \psi\|_1 = R^{|\alpha|} \|D^\alpha \phi\|_1$

(i) $\|D^\alpha f\|_p = \|D^\alpha (f * \psi)\|_p = \|f * (D^\alpha \psi)\|_p \le \|f\|_p \|D^\alpha \psi\|_1$

$= C_\alpha R^\alpha \|f\|_p$; $(C_\alpha = \|D^\alpha \phi\|_1)$

(ii) Let r be s.t. $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ (≥ 0)

$\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}$

so $\frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1$

$\frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1$

then by Young's inequality

$$\|f\|_q = \|f * \psi\|_q \le \|f\|_p \|\psi\|_r = \|f\|_p \cdot R^{n\left(1 - \frac{1}{r}\right)} \|\phi\|_r$$

$$= C R^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_p$$

Note $f \in L^2$, $\hat{f} \in L^\infty$ and $\operatorname{supp} \hat{f} \subseteq B_R \Rightarrow \hat{f} \in L^1$ $(f \in C^\infty) \Rightarrow f \in L^\infty$, $\|f\|_\infty \le C$ □

$\|D_j f\|_\infty \le C R \|\hat{f}\|_1 \Rightarrow \|\nabla f\|_\infty \le C R \Rightarrow |f(x) - f(y)| \le \int_x^y |\nabla f| \underset{\le |y-x| \cdot R}{\overset{= \|\hat{f}\|_1}{}}$

$\int \xi \cdot \hat{f}(\xi)$

Bernstein's ineq. for ellipsoids

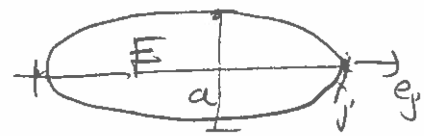
- $E = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{|(x-a)\cdot e_j|^2}{r_j^2} \le 1\}$, $e_1, \dots, e_n$ standard basis vectors in $\mathbb{R}^n$

$\quad = E_a(r_1, \dots, r_n)$ (or any set of orthonormal vectors)

<u>ie.</u> if $a = 0$ then we have $\quad a =$ center, $r_j =$ width's

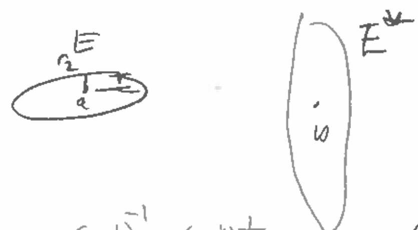
$$\sum_{j=1}^{n} \left(\frac{x_j}{r_j}\right)^2 \le 1 \text{ ; so if } y_j = \frac{x_j}{r_j} \text{ then}$$

$$\sum_{j=1}^{n} y_j^2 \le 1 = B_1$$



- E_a and E_b^* are dual ellipsoids, if

if $r_j^* = r_j^{-1}$ where $E = E_a(r_1, \dots, r_n)$, $E^* = E_a(r_1^*, \dots, r_n^*)$



<u>Prop 11.4</u> (Bernstein's ineq. for an ellipsoid)

Let $f \in L^1 + L^2$, supp $\hat{f} \subseteq E$. Then

$$\|f\|_q \le C\, |E|^{\frac{1}{p} - \frac{1}{q}} \|f\|_p$$

$(T^t)^{-1} = (T^{-1})^t$ $\quad \langle x, y \rangle = \langle x, T^{-1} \dots$

$(T^t)^{-1} T^t = I$ $\quad \langle T^{-1} x, y \rangle$

$x_j \le 1 \to r_j x_j$ $\quad \langle T T^{-1} x, y \rangle = \langle T^{-1} x, \dots$

<u>Pf</u> Let $E = E_k(r_1, \dots, r_n)$, let $T : (x_1, \dots, x_n) \to (r_1 x_1, \dots, r_n x_n)$ $\Rightarrow T^t = T$

$T : B_1 \longmapsto E - k$ lin. trf.

Let $S = T^{-t} \Rightarrow T = S^{-t}$, $f_1 := e^{-2\pi i k \cdot x} f(x)$, $g = f_1 \circ S$

Then $\hat{g}(\xi) = |\det S|^{-1} \hat{f}_1(S^{-t}(\xi)) = |\det S|^{-1} \hat{f}(S^{-t}(\xi) + k)$

$\quad = |\det T|\, \hat{f}(T\xi + k)$

thus $\hat{g}(\xi) \ne 0 \iff \hat{f}(T\xi + k) \ne 0 \iff T\xi + k \in E \iff T\xi \in E - k$

$\quad \iff \xi \in B_1$, Rd

$\Rightarrow \|g\|_q \le \|g\|_p$

By $\|g\|_q = \|f_1 \circ S\|_q = |\det S|^{-\frac{1}{q}} \|f\|_q = |\det T|^{\frac{1}{q}} \|f\|_q = |E|^{\frac{1}{q}} \|f\|_q$

$\Rightarrow |E|^{1/q} \|f\|_q \le |E|^{\frac{1}{p}} \|f\|_p$ $\quad \square$

Finally, we can claim which suggests that f is not changing much on the dual ellipsoid E^*.

Namely, if $\operatorname{supp}\hat{f} \subseteq E$, then: $\sup_{x\in E^*} |f(x)| \le C \frac{1}{|E|^*}\int_{E^*} |f|\,dx$

<u>Prop 11.5</u> Let $f \in L^1 + L^2$, $\operatorname{supp}\hat{f} \subseteq E$. Then $\forall z \in E^*$

$$|f(z)| \le C \frac{1}{|E^*|}\int |f(x)|\,\phi_{E^*}(x)\,dx,$$

<u>Pf</u> $E = E^* = B_1$ (unit ball), Choose $\varphi \in S$ s.t. $f = f * \varphi$.

Then $|f(z)| \le \int |f(x)|\cdot|\varphi(z-x)|\,dx$, let $N \in \mathbb{N}$

$$\le C_N \int |f(x)|\,(1+|z-x|^2)^{-N}\,dx$$

Now if $|z| \le 1$, then for $|x| \le 2$ $\quad 1+|x|^2 \le 5 \le 5\,(1+|z-x|^2)$

if $|x| \ge 2$, then $|z-x| \ge 2$ so

If $|z| \le 1$, then $\quad 1+|x|^2 \le 1+|z-x|^2$

$\Rightarrow$

$$|f(z)| \le C_N \int |f(x)|\,(1+|x|^2)^{-N}\,dx$$

$$\le C_N \int_{|x|\le 1} |f(x)|\,dx$$

$\int_1^\infty r^{n-1}(1+r)^{-N}\,dr$

$\int_1^\infty r^{n-N} \le 1$

$r^{n-1}\, 2^{-\frac{N}{2}}(1+1)^{-\frac{N}{2}} \le 2^{-\frac{N}{2}}$

Let $E^* = E_k(r_1, \dots, r_n)$; define the smoothed indicator function of E^*:

$$\phi_{E^*}(x) := \Big(1 + \sum_{j=1}^{N} \frac{|(x-k)\cdot e_j|^2}{r_j^2}\Big)^{-N}$$

Prop 5.5   Let $f \in L^1 + L^2$; $\operatorname{supp} \hat f \subseteq E$. Then $\forall z \in E^*$

$$|f(z)| \le C_N \underbrace{\frac{1}{|E^*|} \int |f(x)|\, \phi_{E^*}(x)\, dx}_{\text{roughly the average of } f \text{ on } E^*}$$

Pf

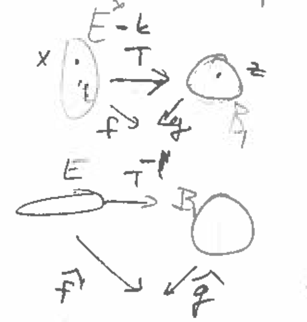
- Let $E = E^* = B_1$, Choose $\varphi \in S$ st. $f = f * \varphi$, then for $|z| \in B_1$

$f = g(T(x-t))$

$g = f$ $g = f(T^{-1} \dots)$

$$|f(z)| \le \int |f(x)||\varphi(z-x)|\, dx \le C_N \int |f(x)| \big(1 + |z-x|^2\big)^{-N} dx$$

$$\le C_N' \int |f(x)| \big(1 + |x|^2\big)^{-N} dx$$



As if $|x| > 2$, then $|x - z| \ge |z| \Rightarrow 1 + |x-z| \ge 1 + |x|$

, but if $|x| < 2$, then $1 + |x|^2 \le 5 \le 5\,(1 + |z-x|^2)$

- $k = 0$ so $E^* = E_k(r_1, \dots, r_k)$, let $g(x) = f(T^{-1}x + k)$? E centered at 0

$$|\hat g(\xi)| = |\det T|\, |\hat f(T\xi)|$$

$\Rightarrow \operatorname{supp} \hat g \subseteq B_1$

$\boxed{T = T^t}$ → diagonal

$\xi \in B_1 \iff T\xi \in E$ so $T : E^{-1} \to B_1 \iff T^{-1} : E^* - k \to B_1$

$\Rightarrow$ for $z \in B_1$; $|g(z)| \le C_N \int |g(x)| \big(1 + |x|\big)^{-N} = \int |g(x)|\, \phi(x)$

$-x \in E^* \Rightarrow z = T(x-k) \in B$ $f(x) = g(z)$, $x = T^{-1}z + k$

$$|f(x)| = \underset{\substack{\| \\ |g(z)|}}{|f(T^{-1}z + k)|} \le \int \underset{|g(x)| = |f(T^{-1}x + k)|}{|f(T^{-1}x + k)|}\, \phi(x) = |\det T| \int |f(z)|\, \phi_{E^*}(z)$$