I. Fourier trf.

$f \in L^1(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} |f(x)|\,dx < \infty$.

Def Let $\xi \in \mathbb{R}^n$, then $\hat{f}(\xi) := \int e^{-2\pi i x\cdot\xi} f(x)\,dx$

If μ is a finite complex meas on $\mathbb{R}^n$; i.e. $\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$

s.t. $\mu\left(\dot{\bigcup}_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ s.t. the RHS abos. conv.

then $\hat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi}\, d\mu(x)$ ← Discuss it just for real/pos. measu

Note • ⇒ $|\mu|(\mathbb{R}^n) = \sup_{\dot{\bigcup}_{i=1}^{\infty} A_i = \mathbb{R}^n} |\mu(A_i)| < \infty$

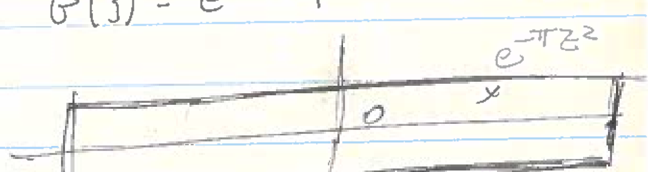
• If $f \in L^1(\mathbb{R}^n)$ then $\mu_f(A) := \int_A f(x)\,dx$ is a compl meas.

Ex $a \in \mathbb{R}^n$; $\delta_a(E) := \begin{cases} 1, & a \in E \\ 0, & a \notin E \end{cases}$ $\Rightarrow$ $\hat{\delta}(\xi) = \int e^{-2\pi i x\cdot\xi}\, \delta_a(x)\,dx$

$= e^{-2\pi i a\cdot\xi}$

Note: $f \in C(\mathbb{R}^n)$ con't $\Rightarrow$ $\int f(x)\,\delta_a(x)\,dx = f(a)$ ← HW

Ex (Gaussian) $G(x) := e^{-\pi|x|^2}$. Then $\hat{G}(\xi) = e^{-\pi|\xi|^2}$



Pf (Complex Analysis) Enough for $n=1$

$$\hat{G}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\cdot\xi} e^{-\pi|x|^2}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2} e^{-\pi\xi^2}\,dx$$

$$= \int_{-i\xi+\mathbb{R}} e^{-\pi z^2}\,dz \cdot e^{-\pi\xi^2} = e^{-\pi\xi^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi t^2}}_{= e^{-\pi\xi^2}}$$

If $z = \pm R + it$

$z^2 = R^2 \pm 2iRt - t^2$

$\mathrm{Re}\, z^2 \geq R^2 - \xi^2$

Prop: $f \in L^1 \Rightarrow$ (1) $\|\hat{f}\|_\infty \le \|f\|_{L^1(\mathbb{R}^n)}$ $(= \|f\|_1)$

(or $\mu \in \mathcal{M}(\mathbb{R}^n) \Rightarrow \|\hat{\mu}\|_\infty \le \|\mu\|_{\mathcal{M}(\mathbb{R}^n)}$)

(2) $\hat{f}$ is con't :

$$\hat{f}(\xi+h) = \int e^{-2\pi i x\cdot\xi} e^{-2\pi i x\cdot h} f(x)\,dx =$$

For every x ~~with~~ we have $\lim_{h\to 0} e^{-2\pi i x\cdot h} = 1$

Let $g_0(x) = e^{-2\pi i x\cdot\xi} f(x)$, $g_h(x) = e^{-2\pi i (x+h)\cdot\xi} f(x)$

Then $\forall x$: $g_h(x) \longrightarrow g(x)$ as $h \to 0$ thus Leb. Dom. Co.

Basic Formulae Let $f \in L^1$

1. $f_\tau(x) = f(x-\tau) \Rightarrow \hat{f_\tau}(\xi) = e^{-2\pi i \tau\cdot\xi}\hat{f}(\xi)$

2. $e_\tau(x) = e^{2\pi i x\cdot\tau} \Rightarrow \widehat{e_\tau f}(\xi) = \hat{f}(\xi-\tau)$

3. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ lin. invertible, then

$$\widehat{f\circ T}(\xi) = \int f(T(x))\, e^{-2\pi i x\cdot\xi}\,dx, \quad y = Tx,\ ~~dx~~\ x = T^{-1}y$$

$$dx = |\det T|^{-1}dy$$

$$= \int f(y)\, e^{-2\pi i T^{-1}y\cdot\xi} |\det T|^{-1} dy = \int f(y) e^{-2\pi i y\cdot\underbrace{(T^{-1})^t\xi}} |\;|^{-1} dy = \_\_$$

$$= |\det T|^{-1}\left(\hat{f}\circ T^{-t}\right)$$

Note • T is orthogonal, i.e. $TT^t = T^tT = I$

then $\widehat{f\circ T} = \hat{f}\circ T$ as $(T^{-1})^t = T$

• f is radial, i.e. $f\circ T = f \iff f(x) = h(|x|)$

then $\hat{f}$ is also radial.

• If $T_\lambda x = \lambda x$ dilation $\Rightarrow \widehat{f \circ T_\lambda} = \lambda^{-n} \hat{f} \circ T_{\lambda^{-1}}$

i.e. if $f_\lambda(x) := f(\lambda x)$ then $\hat{f}_\lambda(\xi) = \lambda^{-n} \hat{f}(\lambda^{-1}\xi)$

Note (Principle) If f is localised in space and converges, then $\hat{f}$ is "smooth" (many times diff.)

Prop A

1. Suppose $f \in C^N$ and $D^\alpha f \in L^1$ for all α with $0 \le |\alpha| \le N$

Then $\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$ for $|\alpha| \le N$, and

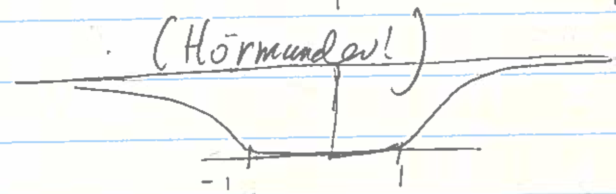
$$|\hat{f}(\xi)| \le C(1+|\xi|)^{-N} \quad (\hat{f} \text{ loc. is space})$$

Test functions Let $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi \in C^\infty$ st

• $\phi(x) = 1$ if $|x| \le 1$ • $\phi(x) = 0$ if $|x| \ge 2$

• $0 \le \phi \le 1$, $\phi(x) = \phi(|x|)$

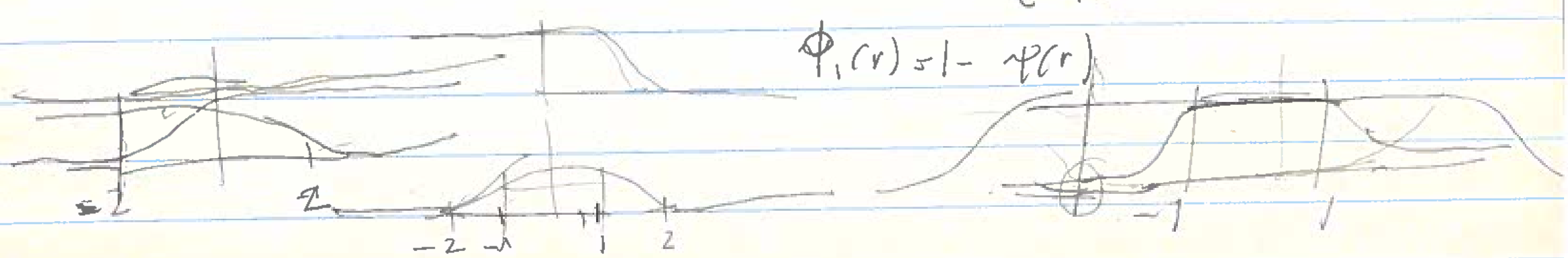
Ex 1. $\psi(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ (Hörmander!)



$\psi_1(r) = \psi(r-1)$ if $r \ge 1$
$= 0$ if $r < 1$

$\psi_{-1}(r) = \psi_1(-r) \Rightarrow \varphi(r) = \begin{cases} \psi_1(r) & \text{if } r > 0 \\ \psi_{-1}(r) & \text{if } r \le 0 \end{cases}$

$\phi_1(r) = 1 - \varphi(r)$



$\otimes$
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Let $\phi_k(x) = \phi\left(\frac{x}{k}\right)$ then $\operatorname{supp} \phi_k \subseteq \{k \le |x| \le 2k\}$

$(D^\alpha \phi_k)(x) = |k|^{-\alpha} (D^\alpha \phi)\left(\frac{x}{k}\right)$ if $|D^\alpha \phi(x)| \le C_\alpha$

$|D^\alpha \phi_k| \le \frac{C_\alpha}{|k|^{-\alpha}}$ unif in k and x $(k>0)$

Lemma (Mollifyers) Supp. $f \in C^N$ and $D^\alpha f \in L^1$ $\forall |\alpha| \le N$.

Let $f_k = \phi_k f$. Then $\forall |\alpha| \le N$ $\| D^\alpha f_k - D^\alpha f \|_1 \to 0$ as $k \to \infty$

Pf: • $\| \phi_k D^\alpha f - D^\alpha f \|_1 = \| (\phi_k - 1) D^\alpha f \|_1 \le \int_{|x| \ge k} |D^\alpha f(x)|\, dx \to 0$ as $k \to \infty$ by LDC

• $D^\alpha(\phi_k f) - \phi_k (D^\alpha f) = \sum_{0 < \beta \le \alpha} c_\beta D^\beta \phi_k \cdot D^{\alpha - \beta} f$

($\beta = 0$ term is exactly $\phi_k \cdot D^\alpha f$)

Now $\| D^\beta \phi_k \|_\infty \le \frac{C_\beta}{|k|^\beta} \le \frac{C}{|k|}$

$\Rightarrow \| R.H.S \|_1 \le \frac{C}{|k|} \sum_{0 < \beta \le \alpha} \| D^{\alpha - \beta} f \|_1 \le \frac{C'}{|k|} \to 0$ $\square$

Pf Lemma A ~~Let~~ Suppose it holds for f_k. i.e.

" $\widehat{D^\alpha f_k} = (2\pi i \xi)^\alpha \hat{f_k}(\xi)$ $\Rightarrow$ $\widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

Then $|\widehat{D^\alpha f} - \widehat{D^\alpha f_k}| \le \| D^\alpha f - D^\alpha f_k \|_1 \to 0$ for $k \to \infty$ $\Rightarrow$

WLOG assume $\operatorname{supp} f$ is compact.

Int. by pa: $\int \frac{\partial f}{\partial x_i}(x) e^{-2\pi i x \cdot \xi} = -\int f(x) \frac{\partial}{\partial x_i}\left(e^{-2\pi i x \cdot \xi}\right) = 2\pi i \xi_i \hat{f}(\xi)$

$\Rightarrow \quad \widehat{D^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

(ii) $\quad D^\alpha f \in L^1 \Rightarrow (2\pi i \xi)^\alpha \hat{f}(\xi) \in L^\infty \Rightarrow \quad \forall |\alpha| \le N$

$\Rightarrow \quad \left(\sum_{|\alpha| \le N} |\xi^\alpha| \right) |\hat{f}(\xi)| \le C_N$

$\Leftrightarrow \quad (1+|\xi|)^N |\hat{f}(\xi)| \le C_N'$

Note: We'll use $\quad 1+|x| \le 1+|y|+|x-y| \le (1+|y|)(1+|x-y|)$

Prop (1.3) Suppose $f \in L^1$ and $\exists R>0$ st $f(x)=0$ $\forall |x| \ge R$.

Then $\hat{f} \in C^\infty$ and $\|D^\alpha \hat{f}\|_\infty \le (2\pi R)^{|\alpha|} \|f\|_1$ bdd

$D^\alpha \hat{f} = ((-2\pi i x)^\alpha f)^\wedge$

Pf: By induction it is enough to prove w/ $|\alpha|=1$

Consider $\alpha = (0 \dots 0, 1, 0 \dots 0)$ $\quad D_j^\alpha g = \frac{\partial g}{\partial x_j} = \lim_{h \to 0} \underbrace{\frac{g(x+he_j)-g(x)}{h}}_{\Delta_j(h) f}$

~~$D_j \hat{f} = \lim$~~ $\quad \Delta_j(h)\hat{f} = \frac{1}{h} \left(\hat{f}(\xi + h e_j) - \hat{f}(\xi) \right) =$

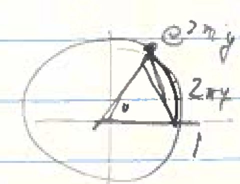
$x \cdot h e_j = h x_j$

$= \int \frac{e^{-2\pi i h x_j} - 1}{h} e^{-2\pi i \xi \cdot x} f(x)\, dx \longrightarrow$

$|e^{2\pi i y} - 1| \le 2\pi |y|$

$\downarrow$

$-2\pi i x_j$ and $\left| \frac{e^{-2\pi i h x_j} - 1}{h} \right| \le \frac{2\pi |h| |x_j|}{|h|} = 2\pi |x_j|$



$\Rightarrow$ L.D.C.

$\longrightarrow \int (-2\pi i x_j) f(x) e^{-2\pi i x \xi} = [(-2\pi i x_j) f]^\wedge(\xi) \quad \square$