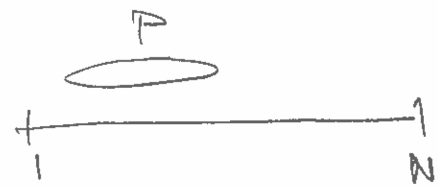
Uniformity

Suppose a set $A \subseteq \mathbb{Z}_N$, $|A| = \alpha N$ is not $\eta = \frac{\alpha^2}{4}$ uniform.

We partitioned $\mathbb{Z}_N$ into progressions P (of equal length) s.t.



$$\frac{1}{N} \sum_P \Big| {\sum_{n \in P}}' (A(n) - \alpha) \Big| \geq \frac{\alpha^2}{8}$$

$\Rightarrow \exists P$ so that $\frac{1}{|P|} \Big| \sum_{n \in P} (A(n) - \alpha) \Big| \geq \frac{\alpha^2}{8}$

$$\frac{1}{N} \sum_P \sum_{n \in P} A(n) - \alpha = \alpha - \alpha = 0$$

Use that $\beta_+ = \frac{1}{2}(|\beta| + \beta)$

$$\Rightarrow \frac{1}{N} \sum_P \Big(\sum_{n \in P} (A(n) - \alpha) \Big)_+ \geq \frac{\alpha^2}{16}$$

$$\Rightarrow \exists P \text{ s.t. } \frac{1}{|P|} \sum_{n \in P} (A(n) - \alpha) \geq \frac{\alpha^2}{16}$$

$$\Rightarrow \frac{|A \cap P|}{|P|} \geq \alpha + \frac{\alpha^2}{16} \leftarrow \text{density increment}$$

Proof of Roth' thm (indirect)

$N \geq \exp\exp(C/\alpha)$ and $A \subseteq [1, N]$ st. $|A| = \alpha N$ & $A \not\ni$ 3-AP's

. Then $\exists P$, $|P| \geq c\,\alpha^2 N^{\frac{1}{2}}$ st. $|A \cap P| \geq (\alpha + \frac{\alpha^2}{16})|P|$

APN/2/5

- $\Phi : P \mapsto [1, N_1]$, $A_1 = \Phi(A) \subseteq [1, N_1]$

$$N_1 = |P| \geq c\,\alpha^2 N^{\frac{1}{2}}$$

$$|A_1| = \alpha_1 N_1, \quad \alpha_1 \geq \left(\alpha + \frac{\alpha^2}{16}\right)$$

- repeating the process we generate

$$A_k \subseteq [1, N_k], \quad |A_k| = \alpha_k N_k; \quad \alpha_k \geq \alpha_{k-1} + \frac{\alpha_{k-1}^2}{16}$$

$$N_k \geq c\,\alpha_{k-1}^2 N_{k-1}^{\frac{1}{2}}$$

- In $k_1 = \frac{16}{\alpha}$ - steps; $\alpha_{k_1} \geq 2\alpha$

In $k_2 = \frac{16}{\alpha} + \frac{16}{2\alpha}$; $\alpha_{k_2} \geq 4\alpha$

In $k \leq \frac{32}{\alpha}$ - steps we get $\alpha_k > 1$ $\lightning$

- Enough to check at $k = 32/\alpha$ $\quad N_k \geq c/\alpha^2$

$$\log N_k \geq \frac{1}{2}\log N_{k-1} - 2\log\left(\frac{1}{\alpha_{k-1}}\right) - c \geq$$

$$\geq \frac{1}{4}\log N_{k-1}$$

$$\log\log N_k \geq \log\log N_{k-1} - \log 4$$

at $k = 32/\alpha$ $\quad \log\log N_k \geq \log\log N - C \geq c/\alpha$

$$\Longrightarrow \log\log N \geq c'/\alpha$$

$$N \geq \exp\exp(c'/\alpha)$$

$\square$

Thm (Behrend's example) $\exists\, A \subseteq [1,N]$

s.t. $|A| \geq N e^{-c\sqrt{\log N}}$ and $A \not\ni$ 3 AP's

Note $N^{1-\varepsilon} \ll |A| \ll N(\log N)^{-k}$ $(\forall \varepsilon, k)$

Proof Let d and k to be choosen later.

Consider the points $\underline{x} = (x_1, \dots, x_d) \in [1,k]^d$.

Then $|\underline{x}|^2 = x_1^2 + \dots + x_d^2 \leq dk^2 \implies \exists\, m \leq kd^2$

s.t. $|S_m| = \{\underline{x};\ |\underline{x}|^2 = m\} \geq k^d / dk^2$.

$$\underline{x} = (x_1, \dots, x_d) \xrightarrow{\ \Phi\ } \Phi(x) = \sum_{i=1}^{d} x_i (2k+1)^{i-1} \in [1, (2k+1)^d]$$

$A = \{\Phi(\underline{x});\ \underline{x} \in S_m\}$, if

$$\Phi(\underline{x}) + \Phi(\underline{y}) = 2\Phi(\underline{z}) \implies \sum_{i=1}^{d} \underbrace{(x_i+y_i)}(2k+1)^{i-1} = \sum_{i=1}^{d} \underbrace{2z_i}(2k+1)^{i-1}$$

$\implies x_i + y_i = 2z_i \ \forall i \implies \underline{x} + \underline{y} = 2\underline{z}$ ↯ $\underline{x}, \underline{y}, \underline{z} \in S_m$

Set

$d = \sqrt{\log N}$, $k = e^{c\sqrt{\log n}}$ $\implies d \log k \approx cN \leq N$

$dk^2 \geq k^2 \geq \underline{e^{2c\sqrt{\log N}}}$

$\square$

<u>Uniformity</u> $\eta>0$, $f:\mathbb{Z}_N \longmapsto \mathbb{C}$, f is η-uniform

if $|\hat{f}(m)| \le \eta \quad \forall m \in \mathbb{Z}_N$

<u>Note</u> A is η-unif $\iff f_A = A - \alpha$ is η-unif

$\forall m \neq 0 \quad |\hat{A}(m)| \le \eta \iff |\hat{f}_A(m)| \le \eta \quad \forall m \iff \|\hat{f}_A\|_\infty \le \eta$

<u>Def</u> $f:\mathbb{Z}_N \longmapsto \mathbb{C}$; $\quad \|f\|_{U^2} = \|\hat{f}\|_{L^4} = \left(\sum_{m\in\mathbb{Z}_N} |\hat{f}(m)|^4\right)^{\frac{1}{4}}$.

<u>Note</u> Let $f:\mathbb{Z}_N \to \mathbb{C}$, $|f| \le 1$.

- If $\|\hat{f}\|_\infty \le \eta \implies \|f\|_{U^2} \le \eta^{\frac{1}{2}}$ ✓

  $\|f\|_{U^2} \le \eta \implies \|\hat{f}\|_\infty \le \eta$ ✓

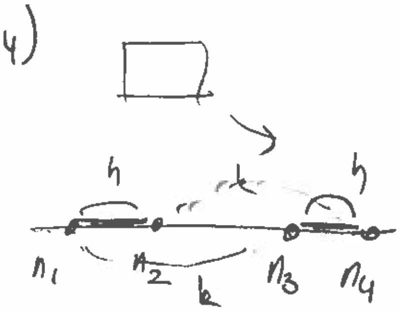
Indeed $\quad \|f\|_{U^2}^4 = \sum_{m\in\mathbb{Z}_N} |\hat{f}(m)|^4 \le \eta^2 \left(\mathbb{E}_n |f(n)|^2\right) \le \eta^2$

<u>Note</u>

$$\|f\|_{U^2}^4 = \sum_m |\hat{f}(m)|^4 = \sum_m \left|\frac{1}{N}\sum_n f(n) e\left(-\frac{m\cdot n}{N}\right)\right|^4 =$$

$$= \frac{1}{N^3} \sum_{n_1,n_2,n_3,n_4} f(n_1)\overline{f(n_2)}\,\overline{f(n_3)}\,f(n_4) \underbrace{\frac{1}{N}\sum_m e\left(-\frac{m(n_1-n_2-n_3+n_4)}{N}\right)}$$

$$= \frac{1}{N^3} \sum_{n_1+n_4=n_2+n_3} f(n_1)\overline{f(n_2)}\,\overline{f(n_3)}\,f(n_4)$$



- $n_2 - n_1 = n_4 - n_3 = h$

  comb. rectangles

ADN 2/2

U^2-uniformity does not control 4-AP's

$N > 2$ prime. $1 > \alpha > 0$.

Let $A \subseteq \mathbb{Z}/N$ s.t. $n \in A \iff n^2 \equiv m \ (N)$ for some $|m| \le \frac{\alpha}{2} N$

$|A| = \alpha N + O(1)$

Let $k \neq 0$, $\delta_{n^2 - m} = \frac{1}{N} \sum_r e\left(\frac{r(n^2-m)}{N}\right)$

$$\hat{A}(k) = \frac{1}{N} \sum_n A(n)\, e\left(-\frac{kn}{N}\right) = \frac{1}{N} \sum_{\substack{|m| \le \frac{\alpha}{2} N \\ n \in \mathbb{Z}_N}} \frac{1}{N} \sum_r e\left(\frac{r(n^2-m) - kn}{N}\right)$$

$$= \frac{1}{N^2} \sum_r \left(\sum_n e\left(\frac{rn^2 - kn}{N}\right)\right)\left(\sum_{|m| \le \alpha N} e\left(-\frac{rm}{N}\right)\right)$$

$$|\hat{A}(k)| \ll \frac{1}{N^2} \sqrt{N} \sum_r \min\left(\alpha N, \frac{1}{\|r/N\|}\right)$$

$\ll N \log N$

$$\ll N^{-\frac{1}{2}} \log N$$

Let $B \subseteq A$, $n \in B \iff n^2 \equiv m \ (N)$ for some $|m| \le \frac{\alpha}{14} N$

B contains $\approx \left(\frac{\alpha}{7}\right)^3 N^2$ 3-AP's

Since $a^2 - 3(a+d)^2 - 3(a+2d)^2 + (a+3d)^2 = 0$

then if $a, a+d, a+2d \in B \Rightarrow (a+3d)^2 \equiv m$, $|m| \le \frac{\alpha}{2} N$

$\Rightarrow a+3d \in A \Rightarrow \{a, a+d, a+2d, a+3d\} \subseteq A$

$\Rightarrow A \gtrsim \alpha^3 N^2$ 4-AP's

$\ggg \alpha^4 N^2$ 4-AP's the expected number of 4-AP's in A!