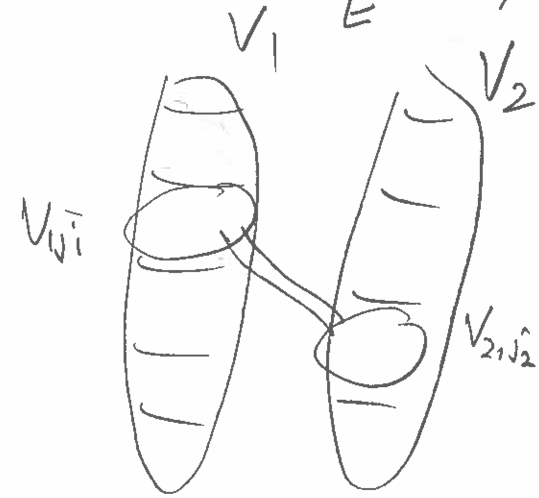
3.5/0



Strong regularity lemma, $\varepsilon > 0$

- $\mathcal{B}_i \subseteq \mathcal{B}_i' \subseteq 2^{\Omega}$; $i=1,2$
  each have at most 2^M atoms
- $J := \lceil 2^M/\varepsilon \rceil$; we constructed partition $V_i = V_{i0} \cup V_{i1} \cup \dots \cup V_{iJ}$ $(i=1,2)$
  s.t. $|V_{i,j}| = \left\lfloor \frac{|V_i|}{J} \right\rfloor$ $\forall i=1,2,\ \forall j \geq 1$
- Given $1 \leq j_1, j_2 \leq J$; $\exists\, \delta \geq 0$ s.t. $\forall A_1 \subseteq V_{1,j_1}, \forall A_2 \subseteq V_{2,j_2}$
  we have $$|E \cap (A_1 \times A_2)| = \delta |A_1||A_2| + O(\varepsilon |V_{1j_1}||V_{2j_2}|) \quad (*)$$
  then (V_{1j_1}, V_{2j_2}, E) is ε-regular

Indeed; $A_1 := V_{1j_1}$, $A_2 = V_{2j_2}$
$$\delta(V_{1j_1}, V_{2j_2}) = \delta + O(\varepsilon)$$

- Let $\delta := \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\big|_{V_{1j_1} \times V_{2j_2}}$ (constant)

$$\frac{1}{|V_1||V_2|} \Big| \sum_{x_1,x_2} \Big(1_E(\underbrace{x_1,x_2}_{x}) 1_{A_1}(x_1) 1_{A_2}(x_2) - \delta 1_{A_1}(x_1) 1_{A_2}(x_2) \Big) \Big| = O\Big(\varepsilon \frac{|V_{1j_1}||V_{2j_2}|}{|V_1||V_2|}\Big)$$

$$|\mathbb{E}_x (1_E - \delta) 1_{A_1} 1_{A_2}| = O(\varepsilon/J^2) \quad (*1)$$

$$|\mathbb{E}_x (1_E - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)) 1_{A_1 \times A_2}| = O(\varepsilon/J^2) \quad (*2)$$

We have by ~~the~~ Thm* (SRL)

$$|\mathbb{E}_x (1_E - \mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2')) 1_{A_1 \times A_2}| = \frac{1}{F(M)}.$$

3,5/2

We have $J = 2^M/\varepsilon$ so if $\frac{1}{F(M)} \le \varepsilon/J^2 = \varepsilon^3/2^{2M}$

i.e. $F(M) := 2^{2M}/\varepsilon^3$ then (⊛2) holds with $\mathcal{B}_i'$ in place of $\mathcal{B}_i$ $(i=1,2)$

So it is enough to show that

$$\left|\mathbb{E}\left(\mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\right) 1_{A_1 \times A_2}\right| = O(\varepsilon/J^2) \quad (⊛3)$$

holds for all but εJ^2 pairs (j_1, j_2) for any

$A_1 \subseteq V_{1j_1}$, $A_2 \subseteq V_{2j_2}$

This follows from

$$\mathbb{E}_x \overbrace{\left|\mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\right|}\cdot \overbrace{1_{V_{1j_1} \times V_{1j_2}}}^{1_{V_{j_1} \times V_{2j_2}}} = O(\varepsilon/J^2) \quad (⊛4)$$

By Thm* prop(2), we have

$$\mathbb{E}_x \left|\mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\right|^2 = O(\varepsilon^3) \quad (⊛5)$$

~~LHS of (⊛4) is bounded~~ by (Cauchy-Schwarz) (⊛4) follows from (⊛6)

$$\underbrace{\mathbb{E}_x}_{\frac{1}{|V_1||V_2|}} \underbrace{\left|\mathbb{E}(1_E \mid --) - \mathbb{E}(1_E \mid --)\right|^2 1_{V_{1j_1} \times V_{2j_2}}}_{\mathrm{Error}(j_1, j_2)} \times \cancel{J^2} = O(\varepsilon^2/J^2)$$

$$\sum_{j_1, j_2}' \mathrm{Error}(j_1, j_2) \le \mathbb{E}_x \left|\mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\right|^2 \le O(\varepsilon^3)$$

$$\Rightarrow \#\{(j_1, j_2) ; \mathrm{Error}(j_1, j_2) \gg \varepsilon^2/J^2\} \le O(\varepsilon J^2)$$

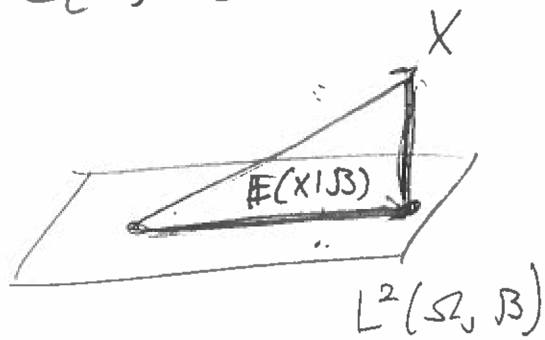
□

Let $X : \Omega \to \mathbb{R}$, $\mathcal{B} \subseteq 2^{\Omega}$ σ-alg. ; $\|X\|_2^2 \leq 1$

$\mathcal{E}(X, \mathcal{B}) = \|\mathbb{E}(X|\mathcal{B})\|_2^2$ "energy" of X on $\mathcal{B}$;

If $\mathcal{B} \subseteq \mathcal{B}'$ then $\mathcal{E}(X, \mathcal{B}') \geq \mathcal{E}(X, \mathcal{B})$

$$\|X\|_2^2 - \mathcal{E}(X, \mathcal{B}) = \|X - \mathbb{E}(X|\mathcal{B})\|_2^2$$



Lemma (lack of regularity $\Rightarrow$ energy increment).

We have $\mathcal{B}_i^* \subseteq 2^{\Omega}$ $(i \in I)$, $\eta > 0$, $X : \Omega \to \mathbb{R}$

$\mathcal{B}^* = \bigvee_{i \in I} \mathcal{B}_i^*$ with $\|X\|_{L^2(\mathcal{B}^*)} \leq 1$.

Let $\mathcal{B}_i \subseteq \mathcal{B}_i^*$ $(i \in I)$ s.t. for some $A_i \in \mathcal{B}_i$

$$\left| \mathbb{E}_x \left(X - \mathbb{E}(X | \bigvee_{i \in I} \mathcal{B}_i)\right) \prod_{i \in I} 1_{A_i} \right| \geq \eta > 0. \quad (I)$$

Then for $\mathcal{B}_i' = \mathcal{B}_i \cup \{\emptyset, A_i, A_i^c, \Omega\}$ we have

$$\mathcal{E}(X | \bigvee_{i \in I} \mathcal{B}_i') \geq \mathcal{E}(X | \bigvee_{i \in I} \mathcal{B}_i) + \eta^2$$

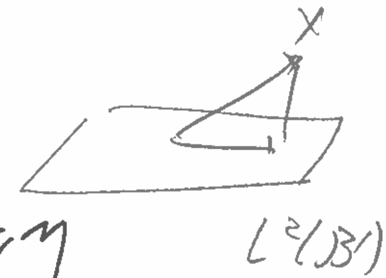
<u>Note</u> $\langle X - \mathbb{E}(X | \bigvee_{i \in I} \mathcal{B}_i), \prod_{i \in I} 1_{A_i} \rangle \leftarrow$ LHS of (I).

Proof

$$\prod_{i\in I} 1_{A_i} = 1_{\bigcap_i A_i} \text{ is } \bigvee_{i\in I} \mathcal{B}_i' =: \mathcal{B}' \text{- measurable}$$

Since $A_i \in \mathcal{B}_i' \subseteq \mathcal{B}'$ for all $i \in I$.

$$\langle X - \mathbb{E}(X \mid \bigvee_i \mathcal{B}_i'), \prod_i 1_{A_i} \rangle = 0$$



$$\Rightarrow |\langle \mathbb{E}(X \mid \bigvee_i \mathcal{B}_i') - \mathbb{E}(X \mid \bigvee_i \mathcal{B}_i), \prod_i 1_{A_i} \rangle| \geq \eta$$

$$\overset{C-S}{\Rightarrow} \| \mathbb{E}(X \mid \bigvee \mathcal{B}_i') - \mathbb{E}(X \mid \bigvee \mathcal{B}_i) \|_2^2 \geq \eta^2$$

$$\overset{\text{Pyth}}{\Rightarrow} \mathcal{E}(X \mid \bigvee_{i\in I} \mathcal{B}_i') - \mathcal{E}(X \mid \bigvee_{i\in I} \mathcal{B}_i) \geq \eta^2$$

Lemma 2. (weak reg. lemma — Frieze-Kannan)

Let $\mathcal{B}_i^0 \subseteq \mathcal{B}_i^* \subseteq 2^\Omega$ for $i \in I$.

Let $X : \Omega \to \mathbb{R}$, $\|X\|_2 \leq 1$, $\eta > 0$.

Then $\exists$ $\mathcal{B}_i^0 \subseteq \mathcal{B}_i \subseteq \mathcal{B}_i^*$ $(i \in I)$ s.t.

(1) $\forall A_i \in \mathcal{B}_i^*$ $(i \in I)$

$$|\langle X - \mathbb{E}(X \mid \bigvee_{i\in I} \mathcal{B}_i), \prod_{i\in I} 1_{A_i} \rangle| \leq \eta$$

(2) $\text{Compl}(\mathcal{B}_i) \leq \text{Compl}(\mathcal{B}_i^0) + O(\eta^{-2})$

Proof Apply Lemma 1 iterated, each time either (1) holds or $1 \geq \mathcal{E}(X | \bigvee \mathcal{B}_i^{(k+1)}) \geq \mathcal{E}(X | \bigvee \mathcal{B}_i^{(k)}) + \eta^2$

so (1) must hold for some $k \leq \eta^{-2}$.

Proof of Thm* (SRL)

Generate $\mathcal{B}_{ik} \subseteq \mathcal{B}'_{ik}$ for $k = 0, 1, \dots$

$k=0$: $\mathcal{B}_{ik} = \mathcal{B}'_{ik} = \{\emptyset, \Omega\}$. $(i \in I)$

$M_k := \max_i \mathrm{Compl}(\mathcal{B}_{ik})$

Such that $\forall i$,

$$|\langle X - \mathbb{E}(X | \bigvee_i \mathcal{B}'_{ik}), \prod_{i \in I} 1_{A_i} \rangle| \leq \frac{1}{F(M_k)} \quad (3)$$

If $\| \mathbb{E}(X | \bigvee_i \mathcal{B}'_{ik}) - \mathbb{E}(X | \bigvee_{i \in I} \mathcal{B}_{ik}) \|_2 < \varepsilon$. (2)

then ✓

If not: set $\mathcal{B}_{i,k+1} = \mathcal{B}'_{i,k}$, let $\eta := \frac{1}{F(M_{k+1})}$

apply Lemma 2, to find $\mathcal{B}'_{i,k+1}$

$$|\langle X - \mathbb{E}(X | \bigvee_i \mathcal{B}'_{i,k+1}), \prod 1_{A_i} \rangle| \leq \frac{1}{F(M_{k+1})}$$

and we have $\mathcal{E}(X | \mathcal{B}_{i,k+1}) - \mathcal{E}(X | \mathcal{B}_{i,k}) \geq \varepsilon^2$

$\Longrightarrow$ (3) & (2) must hold at some stage $k \leq \varepsilon^{-2}$.