Strong Regularity Lemma

Conditional expectation $f : \Omega \to \mathbb{R}$, ($\Omega$ is finite), $\mathcal{B} \subseteq 2^{\Omega}$

f is $\mathcal{B}$-meas. if $f^{-1}(c) \in \mathcal{B}$ $\forall c \in \mathbb{R}$

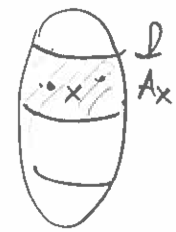
$L^2(\Omega, \mathcal{B}) = \{ f : \Omega \to \mathbb{R},\ f \text{ is } \mathcal{B}\text{-meas} \}$; $\|f\|_2 = \left(\mathbb{E}_x |f(x)|^2 \right)^{\frac{1}{2}}$

($\mathbb{P}$ = normalized counting meas.)

$\mathbb{E}(f \mid \mathcal{B})(x) = \mathbb{E}_{y \in A_x} f(y)$

$\int_{\Omega} f \, d\mathbb{P} = \mathbb{E}_x f(x)$

<u>Note</u> • $\mathbb{E}(f \mid \mathcal{B})\big|_A$ is constant, $\forall A \in \mathcal{B}$ atom

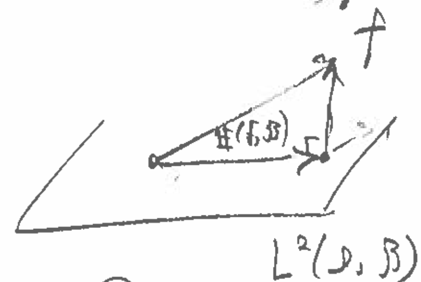


$\Longrightarrow$ $\mathbb{E}(f \mid \mathcal{B})$ is $\mathcal{B}$-meas.

• $\sum_{x \in A} \mathbb{E}(f \mid \mathcal{B})(x) = \sum_{x \in A} f(x)$

<u>Claim</u> $\mathbb{E}(f \mid \mathcal{B})$ is the orthogonal proj. of f to $L^2(\Omega, \mathcal{B})$.

<u>Pf:</u> Let $g \in L^2(\Omega, \mathcal{B})$ and



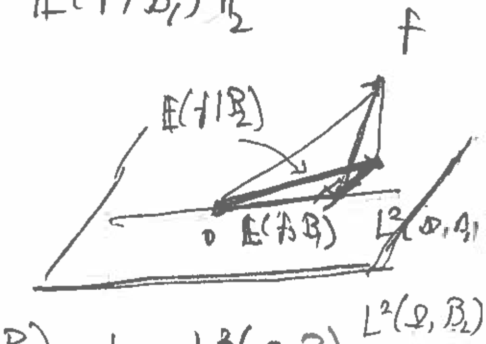
$$\langle f - \mathbb{E}(f \mid \mathcal{B}), g \rangle = \mathbb{E}_x \left(f(x) - \mathbb{E}(f \mid \mathcal{B})(x) \right) \bar{g}(x)$$

$$= \frac{1}{|\Omega|} \sum_{\substack{A \in \mathcal{B} \\ \text{atom}}} \sum_{x \in A} \left(f(x) - \mathbb{E}(f \mid \mathcal{B})(x) \right) \underbrace{\bar{g}(x)}_{\text{constant}} = 0$$

□

Pythagorean Thm Let $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subset 2^{\Omega}$ σ-alg, $f: \Omega \to \mathbb{R}$.

$$\| \mathbb{E}(f \mid \mathcal{B}_2) \|_2^2 = \| \mathbb{E}(f \mid \mathcal{B}_1) \|_2^2 + \| \mathbb{E}(f \mid \mathcal{B}_2) - \mathbb{E}(f \mid \mathcal{B}_1) \|_2^2$$



Pf: f is $\mathcal{B}_1$-meas $\Rightarrow$ f is $\mathcal{B}_2$-meas

$L^2(\Omega, \mathcal{B}_1) \leq L^2(\Omega, \mathcal{B}_2)$

$$\mathbb{E}(f \mid \mathcal{B}_2) - \mathbb{E}(f \mid \mathcal{B}_1) = \big(f - \mathbb{E}(f \mid \mathcal{B}_1)\big) - \big(f - \mathbb{E}(f \mid \mathcal{B}_2)\big) \perp L^2(\Omega, \mathcal{B}_1)$$

$$\| \mathbb{E}(f \mid \mathcal{B}_2) \|_2^2 = \| \underbrace{\mathbb{E}(f \mid \mathcal{B}_1)} + \underbrace{\mathbb{E}(f \mid \mathcal{B}_2) - \mathbb{E}(f \mid \mathcal{B}_1)} \|_2^2 =$$

$$= \| \mathbb{E}(f \mid \mathcal{B}_1) \|_2^2 + \| \mathbb{E}(f \mid \mathcal{B}_2) - \mathbb{E}(f \mid \mathcal{B}_1) \|_2^2 + 0$$

$\square$

Def · $\mathcal{E}(f, \mathcal{B}) = \| \mathbb{E}(f \mid \mathcal{B}) \|_2^2$ energy of f wrt. $\mathcal{B}$ ("index")

· If $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then $\mathcal{E}(f, \mathcal{B}_2) \geq \mathcal{E}(f, \mathcal{B}_1)$

Def: If $\mathcal{B}$ is an alg. on Ω, then its <u>complexity</u> $\mathrm{Compl}(\mathcal{B})$ is the min. number of sets which generates $\mathcal{B}$.

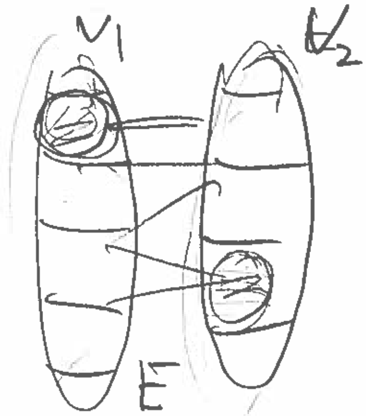
Note If $\mathrm{compl}(\mathcal{B}) = m$, then

$\mathcal{B}$ has $\leq 2^m$ atoms;

($\mathcal{B}$ has k atoms $\Rightarrow$ $\mathrm{Compl}(\mathcal{B}) \sim \log_2 k$)

Def $\mathcal{B}_1, \mathcal{B}_2 \subseteq 2^{\Omega}$ alg; $\mathcal{B}_1 \vee \mathcal{B}_2$ is the alg. gen. by $\mathcal{B}_1$ and $\mathcal{B}_2$.

$$\mathrm{Coupl}(\mathcal{B}_1 \vee \mathcal{B}_2) \leq \mathrm{Coupl}(\mathcal{B}_1) + \mathrm{Coupl}(\mathcal{B}_2)$$



Ex (bipartite graphs) $G = (V_1, V_2, E)$

- $E \subseteq V_1 \times V_2$; $V_1 = V_{11} \cup \dots \cup V_{1m}$, $V_2 = V_{21} \cup \dots \cup V_{2m}$ } partition

$\Omega = V_1 \times V_2$; $\mathcal{B}_1$ is the algebra with atoms

$$A_i = V_{1i} \times V_2$$

$\mathcal{B}_2$ — " ——

$$A_j = V_1 \times V_{2j}$$

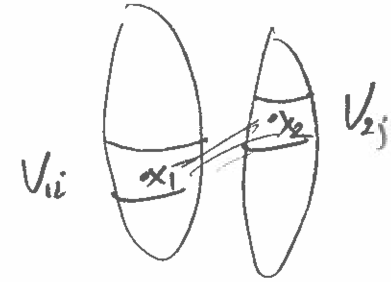
then $\mathcal{B} = \mathcal{B}_1 \vee \mathcal{B}_2$ is the algebra with atoms:

$$A_{ij} = V_{1i} \times V_{2j}$$ (complete bip. graph on V_{1i} and V_{2j})

- $f = 1_E : \Omega \mapsto \mathbb{R}$ is a random var. ;

$$\mathbb{E}(1_E \mid \mathcal{B})(x_1, x_2) = \frac{|E \cap (V_{1i} \times V_{2j})|}{|V_{1i}|\,|V_{2j}|} = \delta_{ij}$$

relative density of edges between V_{1i} and V_{2j}



Thm* (Strong. reg. lemma, prob. version)

Let $(\Omega, \mathcal{B}^*, \mathbb{P})$ be a prob. space, $(\mathcal{B}_i^*)_{i \in I}$ be a finite collection of σ-alg. $\mathcal{B}_i^* \subseteq \mathcal{B}^*$, and let $X : \Omega \mapsto \mathbb{R}$ be a random var., with $\|X\|_{L^2(\mathcal{B}^*)} \leq 1$.

Let $\varepsilon > 0$, $m \in \mathbb{N}$ and $F : \mathbb{R}^+ \to \mathbb{R}^+$ mon. increasing.

Then $\exists$ σ-alg. $\mathcal{B}_i \subseteq \mathcal{B}_i' \subseteq \mathcal{B}_i^*$ for $i \in I$ ④

and $M \in \mathbb{N}$ s.t. the foll. holds

(a) (bound on M); We have $m \le M$, $M \le C(\varepsilon, m, F)$

(b) (complexity bound) $\mathrm{Compl}(\mathcal{B}_i) \le M$

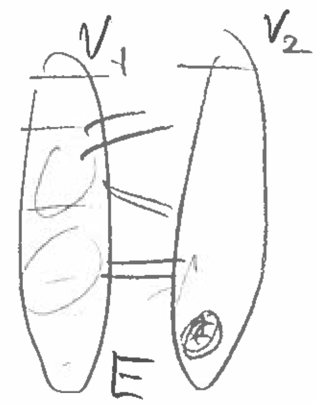
(c) $\left\| \mathbb{E}(X \mid \bigvee_{i \in I} \mathcal{B}_i') - \mathbb{E}(X \mid \bigvee_{i \in I} \mathcal{B}_i) \right\|_2 \le \varepsilon$

(d) (fine approximation is extremely accurate)

For any $A_i \in \mathcal{B}_i^*$ $i \in I$

$$\left| \left(X - \mathbb{E}(X \mid \bigvee_{i \in I} \mathcal{B}_i')\right)(x) \prod_{i \in I} 1_{A_i}(x) \right| \le \frac{1}{F(M)}$$

<u>Proof</u> (Thm* $\Longrightarrow$ Szem's reg. lemma)



$\Omega = V_1 \times V_2$; $I = \{1, 2\}$, $\mathcal{B}_1^* = \{A_1 \times V_2 ; A_1 \subseteq V_1\}$

$\mathcal{B}_2^* = \{V_1 \times A_2 ; A_2 \subseteq V_2\}$

$\mathcal{B}^* = \mathcal{B}_1^* \vee \mathcal{B}_2^* = 2^{\Omega}$, $X = 1_E$

Let $\varepsilon > 0$ and let $F(m) = 2^{2m} \varepsilon^{-3}$; and apply Thm* with $\varepsilon^{3/2}$ in place of ε:

$\exists$ $\mathcal{B}_1 \subseteq \mathcal{B}_1' \subseteq \mathcal{B}_1^*$, $\mathcal{B}_2 \subseteq \mathcal{B}_2' \subseteq \mathcal{B}_2^*$ s.t. $\mathrm{compl}(\mathcal{B}_i) \le M$

where $M = M(\varepsilon)$, s.t.

(a)

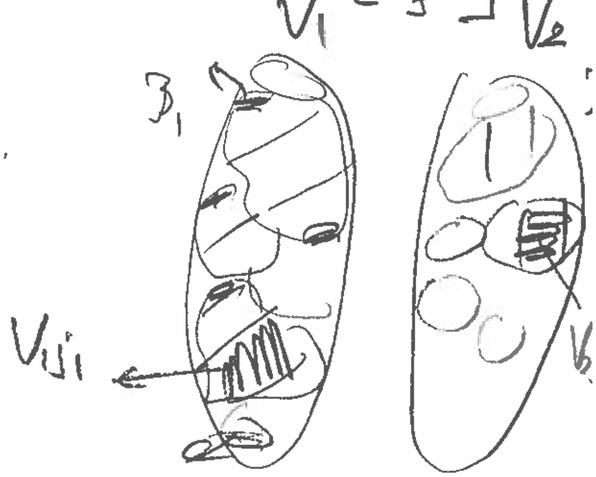
(b) $\| \mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2) \|_2 \le \varepsilon^{3/2}$

(c) $\left| \mathbb{E}_x \left(1_E - \mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2')\right)(x)\, 1_{A_1}(x) 1_{A_2}(x) \right| \le \varepsilon^3/2^{2M} \le$

for any $A_1 \subseteq V_1$, $A_2 \subseteq V_2$

Let J be a large integer $\left(J = \lceil 2^{2M}/\varepsilon \rceil \right)$ and divide each atom of $\mathcal{B}_1$ on V_1 into sets of equal size $\left\lfloor \frac{|V_1|}{J} \right\rfloor$ and an error of size $\le \frac{|V_1|}{J}$.

Do the same partition of $\mathcal{B}_2$ on V_2.



Combining the "errors" into single sets $V_{1,0}$, $V_{2,0}$; we have

$$|V_{i,0}| \le 2^M \frac{|V_1|}{J} \le \varepsilon |V_i| \quad (i=1,2)$$

We have $V_i = V_{i0} \cup V_{i1} \cup \dots \cup V_{iJ}$; $|V_{ij}| = \left\lfloor \frac{|V_i|}{J} \right\rfloor$

- Given a pair j_1, j_2: $G_{j_1 j_2} = (V_{1j_1}, V_{2j_2}, E)$ it is ε-regular if $\forall A_1 \subseteq V_{1j_1}$, $\forall A_2 \subseteq V_{2j_2}$

$$|E \cap (A_1 \times A_2)| = \delta_{j_1 j_2} |A_1 \times A_2| + O(\varepsilon |V_{1j_1}| |V_{2j_2}|) \quad (1) \quad \frac{1}{|V||V_2|}$$

where $\delta_{j_1 j_2} = \dfrac{|E \cap (V_{1j_1} \times V_{2j_2})|}{|V_{j_1}| |V_{j_2}|}$ is the density of E on V_{1j_1} and V_{2j_2}

$$= \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)$$

HW 3.3

(6)

$$|E \cap (A_1 \times A_2)| = \sum_{(x_1, x_2) \in \Omega} 1_E(x_1, x_2)\, 1_{A_1 \times A_2}(x_1, x_2)$$

$$\mathbb{E}_{x \in \Omega}\, 1_E(x)\, 1_{A_1 \times A_2}(x) = \mathbb{E}_{x \in \Omega}\, \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)(x)\, 1_{A_1 \times A_2}(x)$$

So (1) is equivalent to

$$\left| \mathbb{E}_{x \in \Omega} \left(1_E - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2)\right)(x)\, 1_{A_1 \times A_2}(x) \right| \leq O(\varepsilon / J^2) \qquad (2)$$

But we have that

$$\mathbb{E}_{x \in \Omega} \left(1_E - \mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') \right)(x)\, 1_{A_1 \times A_2}(x) \,|= \varepsilon^3 / 2^{2M} \underset{=\frac{1}{F(M)}}{\leq} \varepsilon / J^2 \qquad (3)$$

(as $J = 2^M/\varepsilon$)

$\downarrow$

enough to show!

$$\cancel{\mathbb{E}_{x \in \Omega}} \left| \langle \mathbb{E}(1_E \mid \mathcal{B}_1' \vee \mathcal{B}_2') - \mathbb{E}(1_E \mid \mathcal{B}_1 \vee \mathcal{B}_2),\ 1_{A_1 \times A_2} \rangle \right| \leq O(\varepsilon / J^2)$$

for all but εJ^2-pairs $1 \leq j_1, j_2 \leq J$ $\Leftarrow$ (b)