Combinatorial approach to Szemeredis' thm

Roth' Thm 1. Let $A \subseteq [N]$, $|A| \geq \alpha N$. Then $\exists\, c(\alpha) > 0$ s.t. A contains at least $c(\alpha) N^2$ 3-progr.

Cor: $N \geq N(\alpha) \Rightarrow A$ cont a non-triv 3-AP's. Pf: $c(\alpha)N^2 > N$ if $N > c(\alpha)^{-1}$.

Note Let $N' \in \mathbb{P}$ s.t. $2N < N' \leq 4N$; then by shifting A we may assume $A \subseteq \left(\frac{N'}{2}, N'\right)$ and $A \geq \alpha' N'$ with $\alpha' \geq \alpha/4$.

If $x, y, z \in A$ and $x + y - 2z \equiv 0 \pmod{N'}$ then $x + y - 2z = 0$.

Roth Thm' Thus it is enough to show that $A \subseteq \mathbb{Z}_N$, $|A| \geq \alpha N$ then A contains $\geq c(\alpha) N^2$; $\mathbb{Z}_N$-progr. of length 3.

Combinatorial approach Let $G_A = (V_1, V_2, V_3, E_A)$ be a 3-partite graph; with $V_1 = \mathbb{Z}_N$, $V_2 = \mathbb{Z}_N$, $V_3 = \mathbb{Z}_N$, $A \subseteq \mathbb{Z}_N$

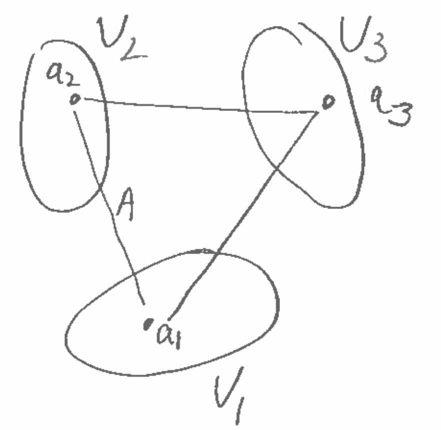
$G_A =$

Let $(a_1, a_2) \in E(V_1, V_2)$ iff $a_2 - a_1 = x \in A$

$(a_2, a_3) \in E(V_2, V_3)$ iff $a_3 - a_2 = y \in A$

$(a_1, a_3) \in E(V_1, V_3)$ iff $a_3 - a_1 = 2z \in 2A$

$(2A = \{2z;\ z \in A\})$



We say that (a_1, a_2, a_3) is a Δ ("triangle") if $(a_1, a_2), (a_2, a_3), (a_1, a_3) \in E$ write $(a_1, a_2, a_3) \in \Delta(E)$

If $(a_1, a_2, a_3) \in \Delta(E) \Rightarrow x + y = 2z$ where $x = a_2 - a_1$, $y = a_3 - a_2$

$2z = a_3 - a_1 = a_3 - a_2 + a_2 - a_1$

Roth Thm 2 $|\Delta(E_A)| \geq c(\alpha) N^3$; i.e. the 3-partite graph contains $\geq c(\alpha) N^3$ triangles

Pf (Thm2 $\Rightarrow$ Thm1')

If $x+y=2z$ with $x,y,z \in A$, then

$\#\{(a_1,a_2,a_3);\ a_2-a_1=x,\ a_3-a_2=y\} = N$ as if $a_2=a_1+X$, $a_3=a_2+y=a_1+x+y$
$a_3-a_1=2z$ so a_1 is free.

$|\{x,y,z\in A;\ x+y=2z\}| = N^{-1}|\Delta(E_A)| \geq c(\alpha)N^2$ $(a_3=a_1+x+y=a_1+2z)$

$\square$

Note If $x=y=z\in A$ then $\forall a_1 \in V_1=\mathbb{Z}_N$ $a_2=a_1+X$, $a_3=a_1+2X$

ie $(a_1, a_1+x, a_1+2x) \in \Delta(E) \Rightarrow$ ~~$\Delta(E)$~~

E contains $\geq \alpha N^2$ edge-disjoint triangles

Indeed (a_1,a_1+x) ~~disjn~~ $\neq (b_1,b_1+y)$ if $a_1\neq b_1$ or $x\neq y$
$(a_1+x,a_1+2x) \neq (b_1+y,b_1+2y)$ e.t.c.

Thm (Δ-removal lemma) Let $G=(V_1,V_2,V_3,E)$ be a 3-partite graph, and let $\alpha>0$, with $|V_1|\sim|V_2|\sim|V_3|\sim N$. If G contains at least αN^2 - edge disjoint Δ's, then G must contain $\geq c(\alpha)N^3$ Δ's i.e. $|\Delta(G)|\geq c(\alpha)N^3$.

Note $c(\alpha)$ is ~~tower~~ very small wrt α; i.e

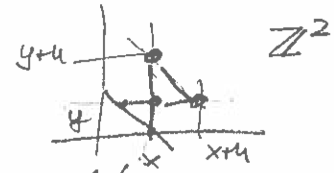
$$c(\alpha)^{-1} \geq \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{c/\alpha \text{ (or } c\log(1/\alpha))}$$

(Szem.) (Fox.)

Problem Is there are non-tower type bound

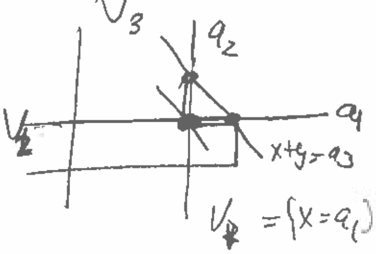
Corner's Thm3 (simplest case of 2-dim Szem' thm)

A corner is a set $\Delta = \{(x,y),(x+u,y),(x,y+u)\}$



Thm (Sz.-A., So.) let $A \subseteq [1,N]^2$, $|A| \geq \alpha N^2$

then A contains $\geq c(\alpha) N^3$ corners.



Pf WLOG $A \subseteq \mathbb{Z}_N^2$; Let $V_1 = V_2 = V_3 = \mathbb{Z}_N^{\{y=a_1\}} = V_1$

Identify: $a_1 \in V_1$ with the line $y = a_1$

$a_2 \in V_2$ — " — $x = a_2$

$a_3 \in V_3$ — " — $x+y = a_3$

then (a_1,a_2) with $l_{a_1} \cap l_{a_2} \in E_A(V_1,V_2) = (a_1,a_2)$

(a_1,a_3) with $l_{a_1} \cap l_{a_3} = (a_3 - a_1, a_1)$

(a_2,a_3) $l_{a_2} \cap l_{a_3} = (a_2, a_3 - a_1)$

If $(a_1,a_2,a_3) \in \Delta_E$ ~~then~~ $\Rightarrow$ gives a corner

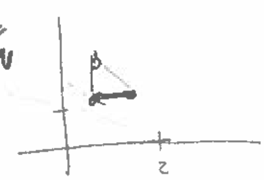
If $\forall (a_1,a_2) \in A$ corresponds to a trivial corner $\Rightarrow (a_1, a_2, a_1+a_2) \in \Delta_E$

$\Rightarrow$ E_A contains $\geq \alpha N^2$ edge disjoint Δ's.

$\Rightarrow$ E_A contains $\geq c(\alpha) N^3$ triangles $\Rightarrow$ A contains $\geq c(\alpha) N^3$ corners

Note Thm3 $\Rightarrow$ Thm2. Let $\phi(x,y) = x + 2y : \mathbb{Z}_N^2 \to \mathbb{Z}_N$

then $\phi(\Delta) = P$ 3-AP's.



but $\phi^{-1}(z) = N$ and let $A' = \phi^{-1}\{A\}$ $\Rightarrow |A'| \geq \alpha N^2$

A' contains $\geq c(\alpha) N^3$ Δ's $\Rightarrow$ A contains $\geq c(\alpha) N^2$ 3-progr.

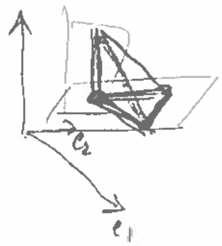
Note This approach generalizes to longer AP's corr. to higher dim. corners.

$\Delta \subseteq \mathbb{Z}_N^3$ is a corner if $\Delta = \{x, x+te_1, x+te_2, x+te_3\}$ where e_1, e_2, e_3 standard basis vectors.

Let $A \subseteq \mathbb{Z}_N^3$, $|A| \geq \alpha N^3$ and define a 4-partite 3-regular hypergraph $G_A = \{V_1, V_2, V_3, V_4, E\}$ with $V_i = \mathbb{Z}_N$; $E = E_{V_1,V_2,V_3} \cup \dots \cup E_{V_2,V_3,V_4}$ and $(a_1, a_2, a_3) \in E_{V_1 V_2 V_3}$ if

$a_4 \in V_4 \Leftrightarrow H_{x_1+x_2+x_3=a_4}$

$a_1 \in V_1 \Leftrightarrow H_{x_1=a_1}$, $a_2 \in V_2 \Leftrightarrow H_{x_2=a_2}$, $a_3 \in V_3 \Leftrightarrow H_{x_3=a_3}$



and $(a_1, a_2, a_3) \in E_{V_1 V_2 V_3}$ if $H_{a_1} \cap H_{a_2} \cap H_{a_3} \in A$

$(a_2, a_3, a_4) \in E_{V_2,V_3,V_4}$ if $H_{a_1} \cap H_{a_2} \cap H_{a_4} \in A$

etc. **Note** $(a_1, a_2, a_3, a_4) \in \Delta_3(E) \Rightarrow \Delta(a_1, a_2, a_3, a_4) \subseteq A$

Thm 4' If $G = (V_1, V_2, V_3, E)$ contains $\geq \alpha N^3$ "edge-disjoint" 3-simplices, then G contains $\geq c(\alpha) N^4$ simplices.

Cor1. If $A \subseteq \mathbb{Z}_N^3$, $|A| \geq \alpha N^3$ then A contains $\geq c(\alpha) N^4_3$-corners

Cor2. If $A \subseteq \mathbb{Z}_N$, $|A| \geq \alpha N$ then A contains $\geq c(\alpha) N^2$ - 4-progr

Pf: Cor1 $\Rightarrow$ Cor2. Let $\phi(a_1, a_2, a_3) = a_1 + 2a_2 + 3a_3$

then $\phi(\Delta) = P$ 4-progr-s since $|\Phi^{-1}(x)| = N^2$

every progr. can arise from at most N^2-corners.

$\square$

Szemeredi's Reg. Lemma

Def Let $G=(V_1,V_2,E)$ be a bip. graph, $\varepsilon>0$. Then we say G is ε-regular if $\forall A_1 \subseteq V_1$, $A_2 \subseteq V_2$:

$$|E \cap (A_1 \times A_2)| = |A_1 \times A_2| \times \frac{|E|}{|V_1 \times V_2|} + O(\varepsilon |V_1 \times V_2|) \qquad (2.1)$$

Note • Writing $\delta(E|A_1\times A_2) := \frac{|E\cap(A_1\times A_2)|}{|A_1\times A_2|}$, edge-density between A_1 & A_2

we get $\delta(E|A_1\times A_2) = \delta(E) + O(\varepsilon \frac{|V_1\times V_2|}{|A_1\times A_2|})$

trivial unless $|A_1\times A_2| \geq \varepsilon V_1 \times V_2$

• Implies the prev. ε-reg. definition, with a larger ε; Indeed ~~if $|A_1| \geq \varepsilon^{?}|V_1|, |A_2|$~~ suppose (2.1) holds with ε^3; then for $|A_1|\geq \varepsilon|V_1|$, $|A_2|\geq\varepsilon|V_2|$ we get $\delta(E|A_1\times A_2) = \delta(E) + O(\varepsilon^3\varepsilon^{-2}) = \delta(E)+O(\varepsilon)$

Thm 5 (Szem's reg. lemma — graph-theoretic version)

Let $0<\varepsilon\leq 1$, $G=(V_1,V_2,E)$ bip graph; with $|V_1|, |V_2| \geq N(\varepsilon)$.

Then $\exists$ $J=J(\varepsilon)$ and decompositions

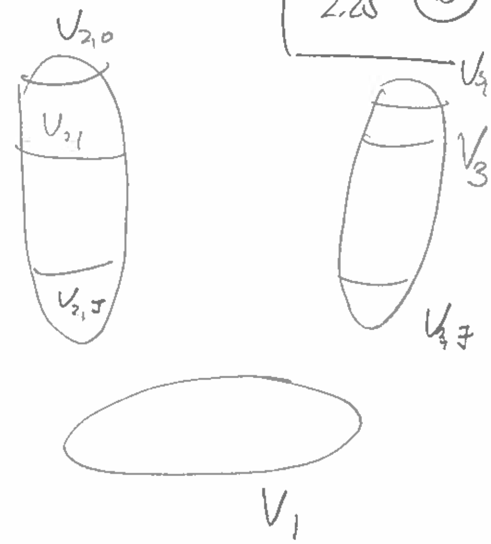
$$V_i = V_{i,0} \cup V_{i,1} \cup \dots \cup V_{i,J} \quad (i=1,2);\ \text{each } s.t.$$

(a) (Except. set) $|V_{i,0}| \leq \varepsilon |V_i|$

(b) (Unif. part.) $|V_{i,j}| = |V_{i,j'}|$ $\forall i=1,2$, $1\leq j\leq J$

(c) (Regularity) The induced bip. graph on (V_{1,j_1}, V_{2,j_2}) is ε-regular for all <u>but</u> $O(\varepsilon J^2)$ pairs $1\leq j_1, j_2 \leq J$.

Proof (Reg. Lemma $\Rightarrow$ Δ-removal lemma)



Suppose $G=(V_1,V_2,V_3,E)$ contains $\geq \alpha N^2$ V_2 edge disjoint Δ's ; with $V_1=V_2=V_3=\mathbb{Z}_N$.

Let $\varepsilon=\varepsilon(\alpha)>0$ to be choosen later ; and apply the Reg. Lemma to $G_{23}=(V_2,V_3,E_{23})$.

We clean up the graph G_{23} as follows:

(1) Remove all edges eminating from $V_{2,0}$ or $V_{3,0}$.
We removed at most $2\varepsilon N^2$-edges.

(2) If $\delta(V_{1,j1}, V_{2,j2})\leq\varepsilon$ then remove all edges between $V_{1,j1}$ and $V_{2,j2}$.
We removed at most $\varepsilon|V_{1,j1}||V_{2,j2}|$-edges from any pair $(V_{1,j},V_{2,j})$
$\Rightarrow$ removed at most εN^2- edges.

(3) If $(V_{1,j1},V_{2,j2})$ not -ε regular , then remove all edges between $(V_{1,j1}, V_{2,j2})$
We removed at most $\varepsilon J^2|V_{1,j1}||V_{2,j2}|\leq\varepsilon J^2\frac{N}{J}\cdot\frac{N}{J}=\varepsilon N^2$ edges

Thus we removed at most $4\varepsilon N^2$-edges.

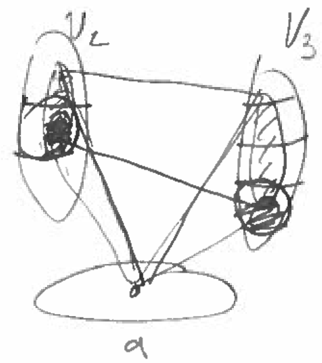
Let $\Delta^*(E)$ the set of edge disjoint Δ's ; and remove all Δ's from $\Delta^*(E)$ if it contain a removed edge.

We removed at most $4\varepsilon N^2\leq\frac{\alpha}{2}N^2$ Δ's from $\Delta^*(E)$.

$\Rightarrow$ the cleaned up graph (V_2,V_3,E'_{23}) contains $\geq\frac{\alpha}{2}N^2$ edge disjoint Δ's.

For $a_1 \in V_1$ let $d(a_1) = \#\{(a_1,a_2,a_3) \in \Delta^*(E')\}$ i.e. no of triangles with one vertex at a_1.

Then $\sum_{a_1\in V_1} d(a_1) \geq \frac{\alpha}{2}N^2 \Rightarrow \#\underbrace{\{a_1 ; d(a_1) \geq \frac{\alpha}{4}N\}}_{W_1} \geq \frac{\alpha}{4}N$



Let $a_1 \in W_1$; $|V_2(a_1)| \geq \frac{\alpha}{4}N$, $|V_3(a_1)| \geq \frac{\alpha}{4}N$,

Let $a_2 \in V_{1,j_1}$, $a_3 \in V_{2,j_2}$ when $(a_1,a_2,a_3) \in \Delta^*(E')$

Then (V_{2,j_1}, V_{3,j_3}) is ε-regular hence

$$|E \cap (V_2(a_1)\cap V_{2,j_2}) \times (V_3(a_1)\cap V_{3,j_3})| \geq$$

$$\geq \frac{\alpha}{2}|V_{2,j_2}\cap V_2(a_1)|\,|V_{3,j_3}\cap V_3(a_1)| - \varepsilon|V_{2,j_2}|\,|V_{3,j_3}|$$

<u>We may need a further clean up:</u>

- Remove all edges between $V_{2,0}$, $V_{3,0}$ and V_1 $\Rightarrow$ $2\varepsilon N^2$-edges.
- If $|V_1(a_1)\cap V_{2,j}| \leq \frac{\alpha}{8}|V_{2,j}|$ remove all edges V_1 and $V_{2,j}$ similarly remove edges between V_1 and V_{3,j_3}.

Total edges remo is at most $\frac{\alpha}{8}N^2 + \frac{\alpha}{8}N^2 = \frac{\alpha}{4}N^2$.

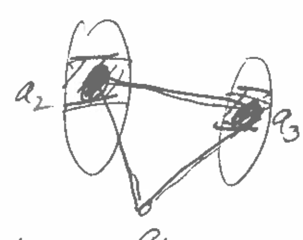
This way we remove at most $\frac{\alpha}{4}N^2 + 2\varepsilon N^2$ edges

E'' has left $\frac{\alpha}{4}N^2 - 2\varepsilon N^2 \geq \frac{\alpha}{8}N^2$ (edges if $\varepsilon \leq \frac{\alpha}{16}$)

Now consider a point: $a_1 \in V_1$, ~~set~~ and a triangle $(a_1,a_2,a_3) \in \Delta^*(E'')$

Let $a_2 \in V_{2,j_2}$, $a_3 \in V_{3,j_3}$; then $|V_2(a_1)\cap V_{2,j_2}| \geq \frac{\alpha}{8}|V_{2,j_2}|$

$|V_3(a_1)\cap V_{3,j_3}| \geq \frac{\alpha}{8}|V_{3,j_3}|$



$$|E\cap(\ \ )\times(\ \ )| \geq \frac{\alpha}{8}\times\frac{\alpha}{8}\times\frac{\alpha}{8}|V_{2,j_2}|\,|V_{3,j_3}| - \varepsilon|V_{2,j_2}|\,|V_{3,j_3}|$$

$\Rightarrow$ there are $\geq c(\alpha)N^2$ Δ's with no ... of $\geq$ ... $\frac{\alpha^3}{\ }|V_{2,j_2}|\,|V_{3,j_3}| \geq \frac{\alpha^3 J(\alpha)^{-1}}{\ } N^2$.