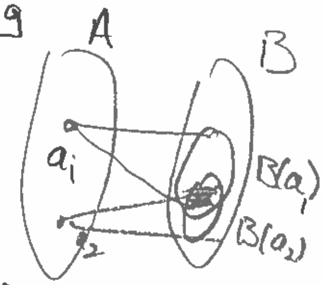
# Qudratic bias

$G$ is a bipartice graph with vertex sets $A, B$;

s.t. $|E| \geq \delta |A||B|$ with given $\delta > 0$. Let $\varepsilon > 0$.

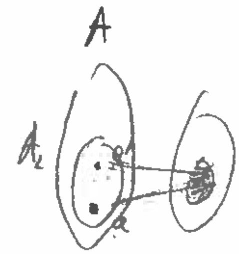



**Lemma 1.** $\exists A_1 \subseteq A$, $|A_1| \geq \frac{\delta}{2}|A|$ s.t.

$$\left|\left\{ (a_1, a_2) \in A_1^2 ;\ |B(a_1) \cap B(a_2)| \leq \frac{\varepsilon\delta}{2} |B| \right\}\right| \leq \varepsilon |A_1|^2$$

**Note** For $\varepsilon$ - a.e. pairs $(a_1, a_2) \in A_1^2$ we have

that $|B(a_1) \cap B(a_2)| > \frac{\varepsilon\delta}{2} |B|$




**Lemma 2.** $\exists A_2 \subseteq A$, $|A_2| \geq \frac{\delta^2}{8} |A|$ s.t.

(1) $\forall a \in A_2$; $|B(a)| \geq \frac{\delta}{2} |B|$

(2) $\forall a \in A_2$; $\left|\left\{ a' \in A_2 ;\ |B(a) \cap B(a')| \geq \frac{\delta^2}{64} |B| \right\}\right| \geq \left(1 - \frac{\delta}{16}\right) |A_2|$

**Note**

- $f : A \to [0,1]$, $\delta > 0$. $\mathbb{E}_{x \in A} f(x) \geq \delta$ $\left( = \frac{1}{|A|} \sum_{x \in A} f(x) \right)$

$$\Rightarrow \left|\left\{ x \in A ;\ f(x) \geq \frac{\delta}{2} \right\}\right| \geq \frac{\delta}{2} |A|$$

- $\mathbb{E}_{x \in A} f(x) \leq \delta$ then

$$\{ |x \in A ;\ f(x) \geq K\delta \} \leq \frac{1}{K} |A|$$

**Pf**

- Let $\bar{A} = \{a \in A;\ |B(a)| \ge \frac{\delta}{2}|B|\} \Rightarrow |\bar{A}| \ge \frac{\delta}{2}|A|$

  ( $\mathbb{E}_{a \in A} |B(a)| \ge \delta$ )

- Use Lemma1, for $(\bar{A}, B)$ with edge density $\ge \frac{\delta}{2}$ and $\varepsilon = \frac{\delta}{32}$

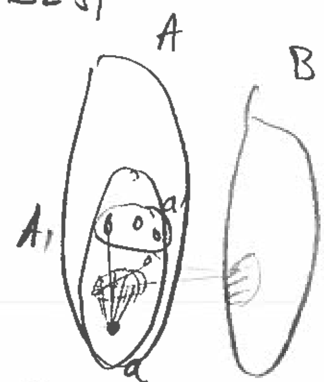
  $\Rightarrow A_1 \subseteq \bar{A}$ s.t. $|\Omega| \le \frac{\delta}{32}|A_1|^2$ where

  $\Omega = \{(a_1, a_2) \in A_1;\ |B(a_1) \cap B(a_2)| \le \frac{\delta^2}{64}|B|\}$

- For $a \in A_1$; $\deg_\Omega(a) = |\{a' \in A_1;\ (a, a') \in \Omega\}|$

  $\mathbb{E}_{a \in A_1} \deg_\Omega(a) \le \frac{\delta}{32}|A_1|$

  $$\Rightarrow |\underbrace{\{a \in A_1;\ \deg_\Omega(a) \ge \frac{\delta}{16}|A_1|\}}_{A_1' \text{ set of "bad elements"}}| \le \frac{1}{2}|A_1|$$




- Let $A_2 := A_1 \setminus A_1'$; $A_2$ satisfies (1) and (2)

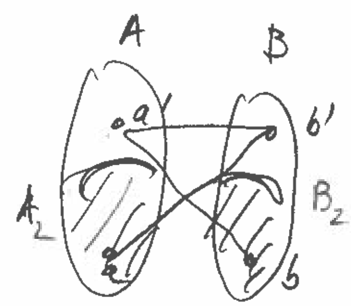
  and $|A_2| \ge \frac{1}{2}|A_1| \ge \frac{1}{2}\frac{\delta}{2}\frac{\delta}{2}|A| \ge \frac{\delta^2}{8}|A|$

**Lemma** $\exists A_2 \subseteq A, B_2 \subseteq B$ s.t. $|A_2| \ge \frac{\delta^2}{8}|A|$, $|B_2| \ge \frac{\delta}{4}|B|$

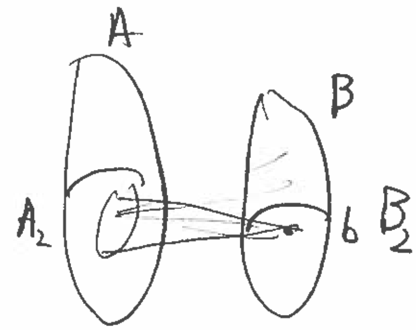
s.t. $a \in A_2, b \in B_2$: $|\{a' \in A, b' \in B:$

$(a, b'), (b', a'), (a', b) \in E\}|$

$\ge \delta^5 |A||B|$



**Proof** Let $A_2$ be given by Lemma 2.



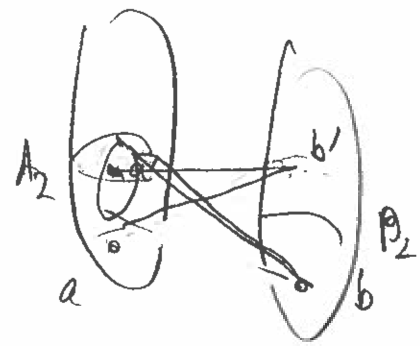

- Let $B_2 = \{ b \in B ;\ |A_2(b)| \geq \frac{\delta}{4} |A_2| \}$.

$$\sum_{b \in B} |A_2(b)| = |E(A_2, B)| = \sum_{a \in A_2} |B(a)| \geq \frac{\delta}{2} |B| |A_2|$$

$\Rightarrow \quad |B_2| \geq \frac{\delta}{4} |A_2|$

- Let $a \in A_2$, $b \in B_2$ and let

$$G(a) = \{ a' \in A_2 ;\ |B(a') \cap B(a)| \geq \frac{\delta^2}{64} |B| \}.$$




By Lemma 2 $\Rightarrow$ $|G(a)| \geq (1 - \frac{\delta}{16}) |A_2|$

$|A_2(b)| \geq \frac{\delta}{4} |A_2|$

$\Rightarrow \quad |G(a) \cap A_2(b)| \geq \frac{\delta}{8} |A_2|$

If $a' \in G(a) \cap A_2(b)$ and $b' \in B(a) \cap B(a')$ then $(b, a'), (a, b'), (a', b') \in E$

Then number of such pairs $(a', b')$ is at least

$$\frac{\delta}{8} |A_2| \times \frac{\delta^2}{64} |B| \geq 2^{-12} \delta^5 |A| |B| \gg \delta^5 |A| |B|$$

$\square$

<u>Note</u> This proves B.-Sz.-G Thm;

by defining $(a,b)\in E$ if $a-b$ is a popular difference;

$$\#\{c,d\in A:\ a-b=c-d\}| \geq \frac{\delta}{2}|A|$$

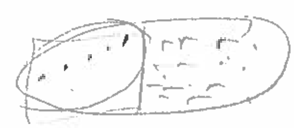
$\Rightarrow$ $A\subseteq \mathbb{Z}_N$ ; $r_4(A)=|\{(a_1,\dots,a_4)\in A^4 : a_1+a_2=a_3+a_4\}| \geq \delta|A|^3$

then $\exists A_2\subseteq A,\ B_2\subseteq A$ with $|A_2|=|B_2|$

s.t. $|A_2-B_2| \ll K(\delta)|A_2|$ $\left(\Rightarrow |A_2-A_2|\ll K(\delta)|A| \text{ or } |A_2+A_2|\ll K(\delta)|A|\right)$

with $K(\delta)\ll \delta^{-16}$

<u>We have</u> $A\subseteq\mathbb{Z}_N$; $|A|=\alpha N$



s.t. $\|f_A\|_{U^3} \geq \frac{\alpha^4}{32}=\eta$ $(f_A=1_A-\alpha)$

- $\exists\ B\subseteq\mathbb{Z}_N$; $\phi: B\to\mathbb{Z}_N$ s.t.

(1) $\forall h\in B$ $|\widehat{\Delta_h f_A}(\phi(h))| \geq \frac{1}{2}\eta^8 = 2^{-6}\alpha^{32}$

(2) Let $\Gamma=\{(h,\phi(h));\ h\in B\}\subseteq \mathbb{Z}_N\times\mathbb{Z}_N$, then

$$r_4(\Gamma)\geq 2^{-5}\eta^{32}N^3 = \underbrace{2^{-165}\alpha^{144}}N^3 =: \delta N^3$$

(3) By B.-Sz.-G., $\exists\ \Gamma'\subseteq\Gamma$ s.t.

$$|\Gamma'|\gg \underline{\underline{\delta}}^5|\Gamma| \quad\text{and}\quad |\Gamma'+\Gamma'|\ll \underline{\underline{\delta}}^{-16}|\Gamma'|$$

$$|\Gamma'|\gg \alpha^C N \quad\text{and}\quad |\Gamma'+\Gamma'|\ll \underbrace{\alpha^{-C}}_{K}|\Gamma'|$$

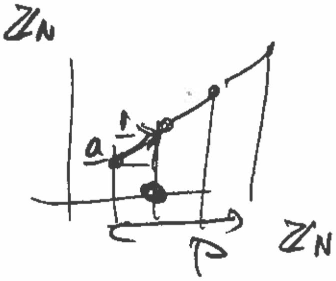
(4) Use Freiman's $\exists Q$ a progression $Q = \{\underline{u} + n\underline{v} \,;\, 1 \le n \le q\}$ $(\underline{u}, \underline{v} \in \mathbb{Z}_N^2)$

s.t. $|\Gamma' \cap Q| \gg \alpha^C |Q|$ and $|Q| \gg \exp(-\alpha^{-C}) N$

$Q = \{(k, \mu + 2\lambda k) \,;\, k \in P\}$,

$P \subseteq \mathbb{Z}_N$ progression of $q = |Q|$ elements

$\Gamma' = \{(k, \phi(k)) \,;\, k \in B'\}$




so. $|B' \cap P| \gg \alpha^C |P|$ and $\forall k \in B' \cap P$

$\phi(k) = \mu + 2\lambda k$

$$\Rightarrow \quad |\widehat{\Delta_k f_A}(\mu + 2\lambda k)| \gg \alpha^C$$

for at least $\alpha^C |P|$ element of $|P|$

<u>Lemma</u> Let $f : \mathbb{Z}_N \mapsto [-1, 1]$, $\lambda \in \mathbb{Z}_N$ s.t.

$$\mathbb{E}_{k \in \mathbb{Z}_N} |\widehat{\Delta_k f}(2\lambda k)|^2 \geq \eta > 0 \,. \quad \Delta_k f(x) = f(x)\overline{f(x+k)}$$

Then $\exists r \in \mathbb{Z}_N$ s.t.

$$\left| \mathbb{E}_{n \in \mathbb{Z}_N} f(n)\, e\left(\frac{\lambda n^2 + rn}{N}\right) \right| \geq \eta^{\frac{1}{2}}$$

<u>Note:</u> $\widehat{\Delta_k f}(2\lambda k) = \mathbb{E}_{n \in \mathbb{Z}_N} \Delta_k f(n)\, e\left(-\frac{2\lambda k \cdot n}{N}\right)$

Pf:

$$\text{LHS} = \mathbb{E}_k \mathbb{E}_{n,h} f(n) f(n+k) f(\underbrace{n+h}_{m}) f(n+h+k)\, e\left(\frac{2\lambda k h}{N}\right)$$

$$2kh = n^2 - (n+k)^2 - (n+h)^2 + (n+h+k)^2$$

let $g(n) = f(n)\, e(\lambda n^2/N)$

$$= \mathbb{E}_{n,h,k}\, g(n)\, \bar{g}(n+k)\, \bar{g}(n+h)\, g(n+h+k) =$$

$$\eta \leq \sum_r |\hat{g}(r)|^4 \leq \left(\max_r |\hat{g}(r)|^2\right)$$

$$\Rightarrow \exists r : |\hat{g}(r)| = \left|\mathbb{E}_n f(n)\, e\left(\frac{\lambda n^2 - rn}{N}\right)\right| \geq \eta^{\frac{1}{2}}$$

$\square$