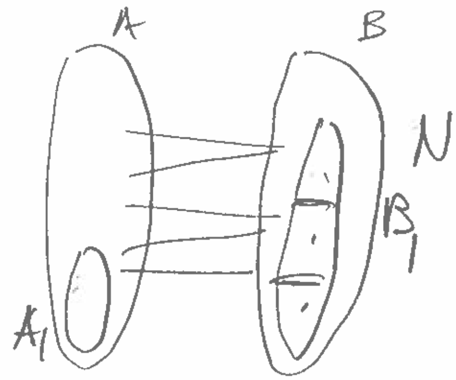
Balog–Szemerédi–Gowers Thm.



G is a bipartite graph with vertex sets A, B and edge set E, $|A| = |B| = N$ s.t. $\delta(G) = \dfrac{|E|}{|A||B|} \geq \delta > 0$.

Lemma 1. Given $\varepsilon > 0$, either G is ε-regular or, $\exists A' \subseteq A$, $B' \subseteq B$ s.t.

$$|A'|, |B'| \geq \frac{\varepsilon^3}{3} N, \quad \frac{1}{2}|B'| \leq |A'| \leq 2|B'|$$

and $$\delta(A', B') = \frac{E(A', B')}{|A'||B'|} \geq \delta + \frac{\varepsilon^3}{6}$$

Proof $\exists A_1, B_1$ s.t. $|A_1|, |B_1| \geq \frac{\varepsilon^3}{6} N$ and

$$\delta(A_1, B_1) \geq \delta + \frac{\varepsilon^3}{3}.$$

Say $|A_1| \leq |B_1|$; let $B_1 = \bigcup^{*}_{i \leq k} B_i$; with $|B_i| \sim |A_1|$ and $k \leq 3/\varepsilon^3$

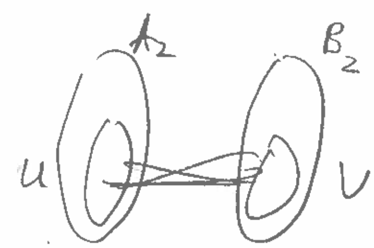
$$\mathbb{E}_{i \leq k}\left(\delta(A_1, B_i) - \delta\right) = \delta(A_1, B_1) - \delta \geq \frac{\varepsilon^3}{6}$$

$$\Rightarrow \exists i \text{ s.t. } \delta(A_1, B_i) \geq \delta + \frac{\varepsilon^3}{6}$$

Lemma 2. $\exists A_2 \subseteq A_1$, $B_2 \subseteq B$ s.t. $|A_2| \sim |B_2|$

$|A_2|, |B_2| \geq \exp(-10\varepsilon^{-3} \log \varepsilon^{-1})$ s.t. (A_2, B_2) is an ε-regular pair, and

$$\delta(A_2, B_2) \geq \delta.$$



$|U| \geq \varepsilon |A_2|$, $|V| \geq \varepsilon |B_2|$

$|\delta(U, V) - \delta(A_2, B_2)| \leq \varepsilon$

(Existence of a regular pair)

Proof By lemma 1 generate $A \supseteq A_1 \supseteq \dots \supseteq A_k$, $B \supseteq B_1 \supseteq \dots \supseteq B_k$

s.t. either (A_k, B_k) is ε-regular or (I)

$$\delta(A_{k+1}, B_{k+1}) \geq \delta(A_k, B_k) + \frac{\varepsilon^3}{6} \quad \text{(II)}$$

$\Rightarrow \exists k \leq \frac{6}{\varepsilon^3}$ s.t. Case (I) happens $\Rightarrow$

(A_k, B_k) is ε-regular, $\delta(A_k, B_k) \geq \delta$, $|A_k| \sim |B_k|$

$$|A_k| \geq N \left(\varepsilon^3/3\right)^k = N\left(\varepsilon^3/3\right)^{6/\varepsilon^3} \gtrsim N \exp(-10\varepsilon^{-3}\log \varepsilon^{-1})$$

$\square$

Thm (B-SZ-G) Let $\delta > 0$. Let $A \subseteq G$ additive group, $|A| = N$

s.t. $\Gamma_4(A) = \{a_1, a_2, a_3, a_4 \in A \,;\, a_1 + a_2 = a_3 + a_4\} \geq \delta N^3$.

Then $\exists\, A', B' \subseteq A$ s.t. $|A'| \sim |B'|$, $|A'| \geq c(\delta) N$

s.t. $|A' - B'| \leq K(\delta)|A'|$, with $c(\delta) = \exp(-10\delta^{-3}\log\delta^{-1})$

and $K(\delta) = \exp(10\delta^{-3}\log\delta^{-1})$

Note By Plünnecke $\Rightarrow$ $|A' - A'| \leq K(\delta)|A'|$ or $|A' + A'| \leq K(\delta)|A'|$

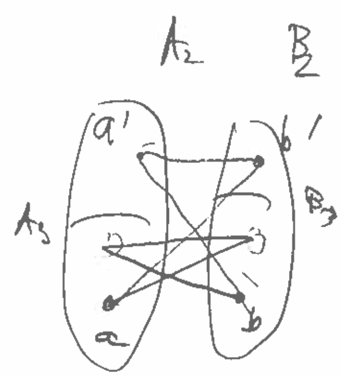
Cor 1. Let A_2, B_2 as in Lemma 2. Then

$\exists\, A_3 \subseteq A_2$, $B_3 \subseteq B_2$; $|A_3| \geq \frac{1}{2}|A_2|$, $|B_3| \geq \frac{1}{2}|B_2|$

then $\forall a \in A_3$, $b \in B_3$

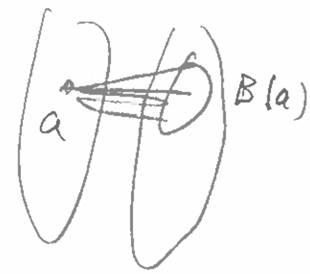
$$N_3(a, b) := \left|\{a' \in A_2, b' \in B_2 \text{ s.t. } (a, b'), (b', a'), (a', b) \in E\}\right|$$

$$\geq \frac{\delta^3}{8} |A_2||B_2|$$



Proof $a \in A_2$, $B_2(a) = \{ b \in B_2 ; (a,b) \in E \}$

$$A_3 := \{ a \in A_2 ; |B_2(a)| \geq \frac{\delta}{2} |B_2| \}$$
$$B_3 := \{ b \in B_2 ; |A_2(b)| \geq \frac{\delta}{2} |A_2| \}$$



since $$\sum_{a \in A_2} |B_2(a)| = |E(A_2, B_2)| \geq \delta |A_2| |B_2|$$

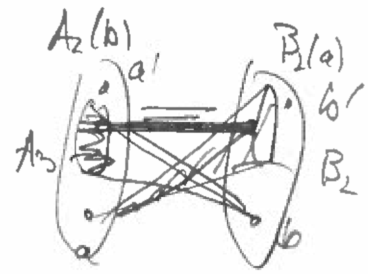
$$\Rightarrow \sum_{a : |B_2(a)| \leq \frac{\delta}{2} |B_2|} |B_2(a)| \leq \frac{\delta}{2} |A_2| |B_2|$$

$$\Rightarrow \sum_{a ; |B_2(a)| \geq \frac{\delta}{2} |B_2|} |B_2(a)| \geq \frac{\delta}{2} |A_2| |B_2| \Rightarrow |A_3| \geq \frac{\delta}{2} |A_2|$$

similarly $|B_3| \geq \frac{\delta}{2} |A_2|$

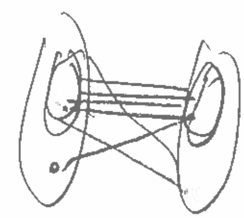
Let $a \in A_3$, $b \in B_3$; $\varepsilon := \frac{\delta}{10}$.

$A_2(b) \geq \varepsilon |A_2|$, $|B_2(a)| \geq \varepsilon |B_2|$



So $|\{ a' \in B_2(a), b' \in A_2(b) ; (a', b') \in E \}|$

$$\geq \frac{\delta}{2} |A_2(b)| |B_2(a)| \geq \frac{\delta^3}{8} |A_2| |B_2|$$



Proof of B-S2. $|A| = N$, $A \subseteq G$.

A pair $(a,b) \in A \times A$ is "popular" if

$$N \geq m(a,b) = |\{ (c,d) \in A \times A ; a - b = c - d \}| \geq \frac{\delta}{2} N$$

$$\sum_{a,b \in A} m(a,b) = |\{ a,b,c,d \in A ; a + d = b + c \}| = r_4(A) \geq \delta N^3$$

$\Rightarrow$ by averaging there are at least $\frac{\delta}{2} N^2$ popular pairs (a,b)

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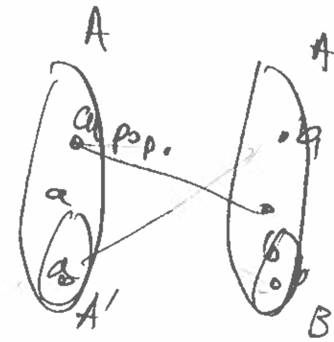
Let $(a,b)\in E$ if (a,b) is popular;



$\delta(G) \geq \frac{\delta}{2}$; $\left(\varepsilon := \frac{\delta}{16}\right)$

by Lemma 3: $\exists\, A'\subseteq A,\ B'\subseteq B$

$|A'|\sim|B'|$, $|A'|,|B'|\geq c(\delta)N$; $\left(c(\delta) = \exp(-10\,\delta^{-3}\log\delta^{-1})\right)$

Let $c = a-b\in A'-B'$, then there are at least

$\frac{\delta^3}{8}|A'||B'|$ pairs (a_1,b_1) s.t. $(a,b_1), (b_1,a_1), (a_1,b)\in E$.

Note $c = a-b = \underbrace{a-b_1}+\underbrace{b_1-a_1}+\underbrace{a_1-b} = x_1-y_1+x_2-y_2+x_3-y_3$ (*)

and $a-b_1 = \underbrace{x_1-y_1}_{\frac{\delta}{2}N}$, $b_1-a_1 = \underbrace{x_2-y_2}_{\frac{\delta}{2}N}$, $a_1-b = \underbrace{x_3-y_3}_{\frac{\delta}{2}N}$

$N_6(c) :=$ total number of choices in (*) for x_1,x_2,x_3,y_1,y_2,y_3

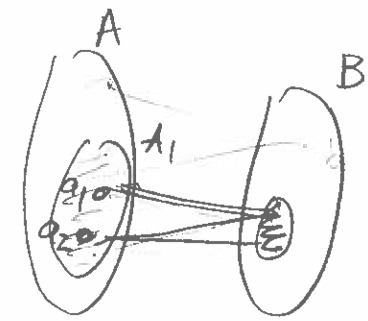
$\Rightarrow$ is at least: $\geq 2^{-6}\delta^6 c(\delta)^2 N^5$.

$$\sum_{c\in A'-B'} N_6(c) \leq N^6 \Rightarrow |A'-B'| \leq \frac{N^6}{2^{-6}c(\delta)^{-2}N^5\delta^6}$$

$|A'-B'| \leq K(\delta)N \leq K'(\delta)|A'|$

(with $K'(\delta)\leq \exp(10\,\delta^{-3}\log\delta^{-1})$)

Gowers proof $\delta > 0$.



Let G be a bip. graph on A, B

s.t. $|E| \ge \delta |A|\,|B|$.

Given $\varepsilon > 0$ $\exists\, A_1 \subseteq A$, $|A_1| \ge \frac{\delta}{2}|A|$ and

$$\left|\left\{a_1,a_2 \in A;\ |B(a_1)\cap B(a_2)| \le \frac{\varepsilon\delta}{2}|B|\right\}\right| \le \varepsilon |A_1|^2$$

Pf:

$$\Sigma_1 = \sum_{a_1,a_2\in A} |B(a_1)\cap B(a_2)| = \sum_{a_1,a_2\in A}\sum_{b\in B} 1_E(a_1,b)\,1_E(a_2,b)$$

$$= \sum_{b\in B}|A(b)|^2 \ge \frac{1}{|B|}\left(\sum_{b\in B}|A(b)|\right)^2 \ge \delta^2|A|^2|B|$$

Let $\Omega = \left\{a_1,a_2\in A;\ |B(a_1)\cap B(a_2)| \le \frac{\varepsilon\delta}{2}|B|\right\}$

then $\Sigma_2 = \sum_{(a_1,a_2)\in\Omega} |B(a_1)\cap B_2(a_2)| \le \frac{\varepsilon\delta^2}{2}|A|^2|B|$

$$\Sigma_1 - \frac{1}{\varepsilon}\Sigma_2 = \sum_b \left(|A_b\times A_b| - \frac{1}{\varepsilon}|A_b\times A_b\cap\Omega|\right) \ge \frac{\delta^2}{2}|A|^2|B|$$

$\exists\, b$: $|\underset{a_1}{A(b)}\times \underset{a_2}{A(b)}\cap\Omega| \le \varepsilon\, |A(b)\times A(b)|$

and $|A(b)\times A(b)| \ge \frac{\delta^2}{2}|A|^2 \Rightarrow |A_b| \ge \frac{\delta}{\sqrt{2}}|A|$

set: $\boxed{A_1 = A(b)}$