2.5 Graphs of Functions

Definition:

f is an even function if \( f(-x) = f(x) \) for every \( x \) in the domain. The graph of an even function is symmetric with respect to the \( y \)-axis, that is, both points \((x,y)\) and \((-x,y)\) are on the graph.

f is an odd function if \( f(-x) = -f(x) \) for every \( x \) in the domain. The graph of an odd function is symmetric with respect to the origin, that is, both points \((x,y)\) and \((-x,-y)\) are on the graph.

Example 1. Determine whether \( f \) is even, odd, or neither even nor odd.

(a) \( f(x) = 3x^4 - 2x^2 + 5 \)
(b) \( f(x) = x^3 + 5 \)

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Given the graph of \( y=f(x) \) and \( c \) is a positive real number, how to obtain the graph of the following equations?

<table>
<thead>
<tr>
<th>Equation</th>
<th>Effect on graph</th>
<th>Graphical interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( y = f(x) + c )</td>
<td>The graph of ( f ) is shifted ( c ) units vertically upward</td>
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<tr>
<td>2. ( y = f(x) - c )</td>
<td>The graph of ( f ) is shifted ( c ) units vertically downward</td>
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<tr>
<td>3. ( y = f(x + c) )</td>
<td>The graph of ( f ) is shifted ( c ) units horizontally to the left</td>
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<tr>
<td>4. ( y = f(x - c) )</td>
<td>The graph of ( f ) is shifted ( c ) units horizontally to the right</td>
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<tr>
<td>Equation</td>
<td>Effect on graph</td>
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<tr>
<td>--------------------------</td>
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<tr>
<td>5. $y = -f(x)$</td>
<td>The graph of $f$ is reflected in x-axis</td>
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<tr>
<td>6. $y = f(-x)$</td>
<td>The graph of $f$ is reflected in y-axis</td>
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<tr>
<td>7. $y = cf(x), c &gt; 1$</td>
<td>The graph of $f$ is stretched vertically</td>
<td></td>
</tr>
<tr>
<td>8. $y = cf(x), 0 &lt; c &lt; 1$</td>
<td>The graph of $f$ is compressed vertically.</td>
<td></td>
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<tr>
<td>9. $y = f(cx), c &gt; 1$</td>
<td>The graph of $f$ is compressed horizontally</td>
<td></td>
</tr>
<tr>
<td>10. $y = f(cx), 0 &lt; c &lt; 1$</td>
<td>The graph of $f$ is stretched horizontally</td>
<td></td>
</tr>
</tbody>
</table>

Example 2. If the point $P(3, -2)$ is on the graph of a function $f$, find the corresponding point on the graph of the function $y = 2f(x - 6) + 3$.

Example 3. Let $y = f(x)$ be a function with domain $D = [-4, 7]$ and range $R = [-5, 8]$. Find the domain $D$ and range $R$ for the function $y = -6f(x + 1)$. Assume $f(4) = 8$ and $f(7) = -5$. 
1.4 Quadratic equations
1. Solving the equation by factoring:
Ex1. Solve for x in the equation: \((x + 13)(x - 3) = -60\)

2. Solving the equation by completing squares:
\[(x^2 + bx) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2\]
Ex2. Solve for x in \(9x^2 + 36x - 1 = 0\)

3. Solving the equation by quadratic formula:
Ex 3. Solve for x in the equation \(ax^2 + bx + c = 0\)
Quadratic formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

If \( b^2 - 4ac > 0 \), the equation has two distinct real roots.
If \( b^2 - 4ac = 0 \), the equation has only one real root.
If \( b^2 - 4ac < 0 \), the equation has no real roots.

Ex. 4
Find the point in the first quadrant at which the graph of the quadratic function
\[ f(x) = 2x^2 - 2 \]
intersects the graph of the linear function
\[ g(x) = 4x + 28 \]

Ex. 5
Determine whether each of the following quadratic functions has no real roots, one real root, or two real roots.
(A) \( 6k^2 + 9k + 7 \)
- no real roots
- one real root
- two real roots

Ex. 6
An object is thrown upward. The height, \( h \), in feet, at time \( t \), in seconds, is given by the formula
\[ h = -16t^2 + 192t \]
Determine the time required for the object to reach a height of 40 feet on its way up. Enter an exact answer, not a decimal approximation.
2.6 Quadratic functions

\[ f(x) = ax^2 + bx + c, \quad (a \neq 0) \]

**Theorem:**

By completing the square, the quadratic function \( f(x) = ax^2 + bx + c \) can always be written in the standard form of

\[ f(x) = a(x-h)^2 + k. \]

In this form, the vertex is \((h,k)\) and the axis of symmetry is \(x=h\).

If \( a > 0 \), the parabola opens upward and \( k \) is the minimum value of \( f \).

If \( a < 0 \), the parabola opens downward and \( k \) is the maximum value of \( f \).

\[ h = -\frac{b}{2a}, \]

Here,

\[ k = f(h) = -\left(\frac{b^2 - 4ac}{4a}\right). \]

Example 1. Express \( f(x) \) in the form \( a(x - h)^2 + k \) where \( f(x) = -\frac{3}{4}x^2 + 9x - 30 \)
Example 2. Find the maximum value of the function \( y = (-4x - 5)(3x + 2) \).

Example 3: Find the standard equation \( (y = a(x - h)^2 + k) \) of a parabola that has a vertical axis and satisfies the given conditions. Be sure to write your answer in the specified format.

Vertex (7, -8), x-intercept 2

Example 4: The number of miles \( M \) that a certain automobile can travel on one gallon of gasoline at a speed of \( v \) mi/hr is given by

\[
M = -\frac{1}{30} v^2 + \frac{5}{2} v \quad \text{for} \quad 0 < v < 60
\]

(a) Find the most economical speed for a trip.
(b) Find the largest value of \( M \).
Example 5: A doorway has the shape of a parabolic arch and is 16 feet high at the center and 8 feet wide at the base. If a rectangular box 7 feet high must fit through the doorway, what is the maximum width the box can have? (enter an exact numeric answer)

Example 6. Find the point P on the graph of \( y = \sqrt{x} \) such that the slope of the line through \((1, 1)\) and \(P\) is \(2/5\).
Example 7. A business (Widgets, Inc) forms a linear model of its widget pricing function based on the following information:
Let \( x \) represent the number of widgets sold, and \( p(x) \) the price per widget in dollars. The firm begins by selling \( x = 350 \) widgets at a set price of $50 each. After holding a "sale", the firm proposes that a $10 discount on the price will yield an increase of 40 more widgets sold.
(a) Find the linear pricing function \( p(x) \) based on this information.
\[
p(x) = \]

(b) Based on the fact that the firm wants both sales and the price to be positive, what is the appropriate domain of the pricing function \( p(x) \)?
\[
\text{domain: } \]

(c) What is the revenue function? (revenue is total sales income)
\[
R(x) = \]

(d) What sales price \( p \) yields maximum revenue?
\[
p = \]
Section 2.7. Operations on Functions

\[(f + g)(x) = f(x) + g(x),\]
\[(f - g)(x) = f(x) - g(x),\]
\[(fg)(x) = f(x)g(x),\]
\[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\]

Example 1. Let \(f(x) = \sqrt{x - 7}\) and \(g(x) = x^2 - 8x - 9\).

(a) Give the formula for \(\left(\frac{f}{g}\right)(x)\).

(b) State the domain of \(\left(\frac{f}{g}\right)(x)\) in interval form.

Definition: composition of functions
Given two functions \(f\) and \(g\), the function \(f \circ g\) is defined by

\[(f \circ g)(x) = f(g(x))\]

The domain of \(f \circ g\) is the set of all \(x\) in the domain of \(g\) such that \(g(x)\) is in the domain of \(f\).

Example 2: Consider the functions below.

\[f(x) = \frac{x}{x - 9},\ g(x) = \frac{2}{x}\]

(a) Find \((f \circ g)(x)\) and the domain of \(f \circ g\).

\((f \circ g)(x) = \text{__________}\)

- All real numbers except 2/9
- All real numbers except 0 and 9
- All real numbers
- All real numbers except 0 and 2/9
(b) Find \((g \circ f)(x)\) and the domain of \(g \circ f\).

\[(g \circ f)(x) = \underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\underline{\under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Section 4.1. Inverse Functions

**Definition of One-to-One function:** A function $f$ with domain $D$ and range $R$ is a one-to-one function if either of the following equivalent conditions is satisfied:

1. Whenever $a \neq b$ in $D$, then $f(a) \neq f(b)$ in $R$.
2. Whenever $f(a) = f(b)$ in $R$, then $a = b$ in $D$.

**Horizontal Line Test:** A function is one-to-one if and only if every horizontal line intersects the graph of $f$ in at most one point.

**Definition of Inverse Function:** Let $f$ be a one-to-one function with domain $D$ and range $R$. A function $g$ with domain $R$ and range $D$ is the inverse function of $f$, provided the following condition is true for every $x$ in $D$ and every $y$ in $R$:

$$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

**Theorem on Inverse Functions:** Let $f$ be a one-to-one function with domain $D$ and range $R$. If $g$ is a function with domain $R$ and range $D$, then $g$ is the inverse function of $f$ if and only if both of the following conditions are true:

1. $g(f(x)) = x$ for every $x$ in $D$
2. $f(g(y)) = y$ for every $y$ in $R$

If a function $f$ has an inverse function $g$, we often denote $g$ by $f^{-1}$. Hence, we have

1. Domain of $f^{-1} = \text{range of } f$
2. Range of $f^{-1} = \text{domain of } f$
3. $f^{-1}(f(x)) = x$ for every $x$ in the domain of $f$.
4. $f(f^{-1}(x)) = x$ for every $x$ in the domain of $f^{-1}$.

Also, the point $(a,b)$ is on the graph of $f$ if and only if the point $(b,a)$ is on the graph of $f^{-1}$. Hence the graphs of $f$ and $f^{-1}$ are symmetric about the line $y=x$. 
Example 1, function is given by a table:

Use the table for $f(x)$ compute each expression. Enter NONE if there isn't any value to report.

(a) $f^{-1}(9) = \underline{\hspace{2cm}}$

(b) $f^{-1}(5) = \underline{\hspace{2cm}}$

(c) $f^{-1}(2) = \underline{\hspace{2cm}}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>7</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-1</td>
<td>9</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

For a function that is given by formula, the procedures to find the inverse function for $y=f(x)$ is the following:

1. Solve for $x$ in the equation of $y=f(x)$.
2. Interchange the $x$ and $y$.

**Example 2:** Find the inverse function of $f$.

$$f(x) = \frac{4x + 3}{5x - 2}$$
Example 3:
Let $h = 12 - 2x$. Use $h$, the table, and the graph to evaluate the expression.

(a) $(g^{-1} \circ f^{-1})(7) =$ 

(b) $(g^{-1} \circ h)(5) =$ 

(c) $(h^{-1} \circ f \circ g^{-1})(5) =$

Example 4. The point $(a,b)$ is on the graph of the one-to-one function $y = f(x)$. For each of the following functions, enter the ordered pair that corresponds to the transformation of $(a,b)$. For example, the graph of $y = f(x) + 1$ is obtained by translating the graph of $y = f(x)$ up one unit so the corresponding point on the new graph is $(a,b + 1)$.

(a) new function: $y = f(12x) - 20$ ordered pair:

(b) new function: $y = -f^{-1}(-x)$ ordered pair:

(c) new function: $y = f^{-1}(x) - 8$ ordered pair: