INTRODUCTION

Let $\Delta$ be an indecomposable root system for a semi-simple simply connected algebraic group defined over the field $K = \mathbb{F}_{q}$. Let $X'$ be the root lattice, $X$ the weight lattice, $X^+$ the set of dominant weights, and $W$ the Weyl group of $\Delta$. Let $\mathbb{Z}[X]$ be the group ring (whose canonical basis we write in the form $e(\lambda)$, $\lambda \in X$), and let $\mathbb{Z}[X]^W$ denote the ring of invariants under the natural action of $W$ on $\mathbb{Z}[X]$.

In [3] certain elements of $\mathbb{Z}[X]^W$ were defined and it was shown that they were generically equal to generalizations given by Verma [11] and Jantzen [8] of invariants defined by Hulsurkar in [4]. In this paper we will define a matrix indexed by $X^+$ and whose entries are integers which are closely related to these elements, and from which in fact the latter can be found. We will give some properties of these elements and show how they can be computed. In particular we will show that the usual partial ordering of $X^+$ makes this matrix upper triangular with ones on the diagonal. We will also show how our methods give some easy derivations of results of Jantzen and show how they can be used to compute generic patterns of the decomposition of Weyl modules into irreducibles.

1. THE ELEMENTS $h_{\lambda, \mu}$

Let $\{\alpha_1, ..., \alpha_l\}$ be a choice of simple roots of $\Delta$, and $\Delta^+$ the corresponding set of positive roots. For $\lambda \in X$ define $\chi(\lambda) = \frac{\sum_{\omega \in W} \det(\omega) e(\omega(\lambda + \rho))}{\sum_{\omega \in W} \det(\omega) e(\omega \rho)}$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Then $\chi(\lambda) \in \mathbb{Z}[X]^W$ and if $\lambda \in X^+$
it is the formal character of the $g_\ell$ module with highest weight $\lambda$. The element $\chi((p-1)\rho)$ is called the Steinberg character and denoted by $st$.

Let $Fr$ denote the Frobenius automorphism, so that $e(\lambda)^{Fr} = e(p\lambda)$. Let $\langle , \rangle$ be the $W$-invariant inner product on $X_p = X \otimes \mathbb{F}$. Let $\alpha^\circ = 2\alpha/\langle \alpha, \alpha \rangle$ be the co-root of $\alpha$. Define $X_p = \{\lambda \in X: 0 \leq \langle \lambda, \alpha_i^\circ \rangle < p \text{ for } i = 1, \ldots, l\}$. Any element $\lambda \in X$ can be written uniquely as $\lambda = \lambda_1 + \lambda_0$, where $\lambda_0 \in X_p$, $\lambda_1 \in X$. We shall standardize this notation so that for $\lambda \in X$, $\lambda_1$ and $\lambda_0$ shall always have this meaning, unless stated otherwise.

For $\lambda \in X$, let $s(\lambda) = \sum_{\mu \in W, \lambda} e(\mu)$, the sum being over distinct conjugates of $\lambda$ under $W$. As in [3], if $S$ is a $\mathbb{Z}$ basis (resp. $\mathbb{Z}[X]^W$ basis) for $\mathbb{Z}[X]^W$ and $\chi \in S$, for $\Psi \in \mathbb{Z}[X]^W$ we denote the coefficient of $\chi$ in $\Psi$ by Mult$_S(\chi, \Psi)$ (resp. Mult$_{S/p}(\chi, \Psi)$), leaving out the subscript $S$ if it is understood from the context. We now define the elements with which we are chiefly concerned in this paper.

**Definition.** For $\lambda, \mu \in X^+$, let $h_{\lambda, \mu} = \text{Mult}_S(\chi(p\lambda_1 + (p-1)\rho), s((p-1)\rho - \lambda_0) \chi(\mu))$, where $S = \{\chi(\lambda): \lambda \in X^+\}$.

For $\lambda \in X_p, \mu \in X^+$, let $H_{\lambda, \mu}^{Fr} = \text{Mult}_{S/p}(st, s((p-1)\rho - \lambda_0) \chi(\mu))$, where $S = \{\chi(\lambda): \lambda \in X_p\}$. (Note that we have denoted this latter element by $h_{\lambda, \mu}$ in [3].)

It follows easily from Weyl's character formula that $\chi(p\lambda + (p-1)\rho) = \chi(\lambda)^{Fr} \chi((p-1)\rho) = \chi(\lambda)^{Fr} st$. Thus an equivalent formulation for the definition of $h_{\lambda, \mu}$ is $h_{\lambda, \mu} = \text{Mult}(\chi(\lambda))^{Fr} st, s((p-1)\rho - \lambda_0) \chi(\mu))$. We also have $H_{\lambda, \mu} = \sum_{\beta \in X^+} \text{Mult}(\chi(\beta))^{Fr} st, s((p-1)\rho - \lambda_0) \chi(\mu)) \chi(\beta) = \sum_{\beta \in X} h_{\lambda + p\beta, \mu} \chi(\beta)$.

We define a collection of new actions of $W$ on $X_p$. For any $\beta \in X_p$, define the **star action with respect to $\beta$** by $\sigma * \alpha = \sigma(\alpha - \beta) + \beta$. Thus the star action is the action of $W$ with the origin shifted to $\beta$. Although this action depends on $\beta$, we do not specify it in the notation. Whenever we write $\sigma * \lambda$ without saying explicitly with respect to which $\beta$ we are taking it, we mean it to be taken with respect to $p(\lambda_1 + \rho) - \rho$.

In case $\beta = -\rho$ the star action is called the dot action and we write $\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$. Let $\tilde{W}$ (resp. $\tilde{W}'$) denote the group generated by $W$ acting with the dot action together with all translations by elements of $pX$ (resp. $pX'$). Call a fundamental domain for the action of $\tilde{W}'$ on $X_{\rho}$ an alcove. We now state our main result concerning the elements $h_{\lambda, \mu}$.

**Theorem 1.** Suppose $h_{\lambda, \mu} \neq 0$. Then there exist $\sigma, \tau \in W$ such that $\sigma \ast \lambda = \tau \cdot \mu$, and $h_{\lambda, \mu} = \sum \text{sgn } \tau$, the sum being over all pairs $(\sigma \ast \lambda, \tau)$ satisfying this condition. Moreover, $\lambda$ and $\mu$ are $\tilde{W}'$-conjugate, and if $\lambda', \mu'$ are $\tilde{W}'$-conjugate and in the interior of the same alcoves as $\lambda$ and $\mu$, respec-
tively, then $h_{\lambda',\mu'} = h_{\lambda,\mu}$. If $\lambda$ is in the interior of an alcove, then $\tau$ and $\sigma$ are uniquely determined so $h_{\lambda,\mu} = \pm 1$.

It follows from the theorem that $h_{\lambda,\mu}$ may be calculated by seeing when an image of an alcove under the dot action intersects with another under the star action. In addition, there is a "translation principle" for the $h_{\lambda,\mu}$, i.e., the value of $h_{\lambda,\mu}$ for $\lambda$ and $\mu$ $\tilde{W}'$-conjugate depends only on the alcoves to which $\lambda$ and $\mu$ belong. Thus we can in fact define $h_{\lambda,\mu}$ for $\lambda, \mu \in X_\tau$.

Before proving Theorem 1 we recall a simple formula due to Brauer.

**Lemma 1.** For any $v, \mu \in X$, $s(v)\chi(\mu) - \sum_{\eta \in W_v} \chi(\mu + \eta)$, the sum being over all distinct conjugates $\eta$ of $v$.

**Proof.** We have

$$s(v)\chi(\mu) = \sum_{\eta \in W_v} e(\eta) \cdot \left( \sum_{w \in W} e(w(\mu + \rho)) \right) \left( \sum_{w \in W} e(wp) \right)$$

$$= \sum_{\eta \in W_v} e(w(\mu + \rho) + \eta) \left( \sum_{w \in W} e(wp) \right)$$

$$= \sum_{\eta \in W_v} e(w(\mu + \rho + \eta)) \left( \sum_{w \in W} e(wp) \right) = \sum_{\eta \in W_v} \chi(\mu + \eta),$$

proving the lemma.

We also note that if $\mu + \rho$ is not $W$-regular, then $\chi(\mu) = 0$, while if it is $W$-regular and $\tau(\mu + \rho) \in X^+$, then $\chi(\mu) = \det \tau \chi(\tau \cdot \mu)$.

**Proof of Theorem 1.** By the definition of $h_{\lambda,\mu}$ and the remark following it, we have $h_{\lambda,\mu} = \sum \det w$, the sum being over all distinct pairs $(w, \eta) \in W \times X$ such that $w(p(\lambda_1 + \rho)) - \rho = \eta + \mu$, where $\eta \in W((p-1)\rho - \lambda_0)$. Suppose $(w, \eta)$ is such a pair with $\eta = w_1((p-1)\rho - \lambda_0)$. Then setting $\tau = w^{-1}$, $\sigma = w^{-1}w_1$ we have $\sigma \ast \lambda = \tau \cdot \mu$. Thus $h_{\lambda,\mu} = \sum \det \tau$, the sum being over all pairs $(\tau, \sigma \ast \lambda)$ satisfying $\sigma \ast \lambda = \tau \cdot \mu$. Now $\tau \cdot \mu$ is certainly $\tilde{W}'$-conjugate to $\mu$, while $\sigma \ast \lambda = \sigma \cdot \lambda + p(\lambda_1 + \rho - \sigma(\lambda_1 + \rho))$ is also $\tilde{W}'$-conjugate to $\lambda$ (since $\lambda_1 + \rho - \sigma(\lambda_1 + \rho)$ lies in the root lattice; so if $h_{\lambda,\mu}$ is nonzero, $\lambda$ and $\mu$ are $\tilde{W}'$-conjugate. If this is the case and if $\lambda'$ and $\mu'$ are as in the statement of the theorem, then $\sigma \ast \lambda' = \tau \cdot \mu'$ whenever $\sigma \ast \lambda = \tau \cdot \mu$, so $h_{\lambda',\mu'} = h_{\lambda,\mu}$. Since the maps $\lambda \rightarrow \sigma \ast \lambda$ and $\mu \rightarrow \tau \cdot \mu$ are both in $\tilde{W}'$, then if $\sigma \ast \lambda = \tau \cdot \mu$ and $\lambda$ (and hence $\mu$) is in the interior of an alcove, these maps are uniquely determined as elements of $\tilde{W}'$, and hence $\sigma$ and $\tau$ are uniquely determined as elements of $W$. 
We will now show that when the partial order induced by strong linkage is put on $X_\beta$, the matrix $h$ becomes triangular. Recall [1, 9, that $\lambda$ and $\mu$ are said to be strongly linked if there is a sequence $\{v_0, \ldots, v_n\}$ with $\lambda = v_0$, $\mu = v_n$, and $v_{i+1} = \sigma_{\alpha_i} v_i$, where $\alpha \in \Delta^+$ (and depends on $i$), and where $\sigma_{\alpha_i} v_i$ is the reflection in $\langle x + \rho, \alpha_i \rangle = np$, and such that $\langle v_{i+1} + \rho, \alpha_i \rangle \geq np$. If the previous conditions are satisfied we write $\lambda \uparrow \mu$. The result follows easily from

**Lemma 2.** Suppose the star action is taken with respect to $\beta$ in $X_\beta$. If $\lambda$ is in the dominant quarter of $X_\beta$ with respect to $\beta$, (i.e., if $\langle \lambda - \beta, \alpha_i \rangle \geq 0$ for all $i$), then for any $\sigma \in W$ there is a sequence $\lambda = v_0, \ldots, v_i = \sigma \ast \lambda$ such that for each $i$ $v_i = \sigma_{\gamma_i} v_{i-1}$ for some $\gamma_i \in \Delta$ and $\langle v_i, \gamma_i \rangle < \langle \beta, \gamma_i \rangle$.

**Proof:** Let $\sigma = \sigma_i \cdots \sigma_1$ be a reduced expression for $\sigma$ in terms of simple reflections with $\sigma_i = \sigma_{\gamma_i}$, $\gamma_i$ a simple root and let $v_i = \sigma_i \cdots \sigma_1 \ast \lambda$. To prove the lemma we only have to show that $\langle \sigma_i \cdots \sigma_1 (\lambda - \beta), \gamma_i \rangle \leq 0$. But $\sigma = \sigma_i (\gamma_i)$ is in $\Delta^-$ (see [5, Corollary 10.2C]), and so the result follows since $\lambda - \beta \in X^+_{\beta}$.

**Corollary.** In the hypothesis of Lemma 2, suppose also that $\beta$ is of the form $p\gamma - \rho$, $\gamma \in X^+$. Then $\sigma \ast \lambda \uparrow \lambda$.

**Proof:** In this case $\sigma \ast x = x - (\langle x + \rho, \alpha_i \rangle - \rho(\gamma, \alpha_i)) \ast \lambda$, so the result follows immediately upon taking $n = \langle \gamma, \alpha_i \rangle$ in the definition of strong linkage.

**Corollary to Theorem 1.** Suppose $\lambda, \mu \in X^+$. If $h_{\lambda, \mu}$ is nonzero, then $\lambda \uparrow \mu$.

**Proof:** By the corollary to Lemma 2, $\tau \cdot \mu \uparrow \mu$. Since $\lambda$ is in the opposite quarter to the dominant quarter with respect to $\beta = p\lambda + (p-1)\rho$, $\sigma_0 \ast \lambda$ is in the dominant quarter (star action with respect to $\beta$) where $\sigma_0$ is the longest element of $W$. Thus $\sigma_0 \ast \lambda \uparrow \sigma_0 \ast \lambda$, and so $\lambda \uparrow \sigma \ast \lambda$. Hence $\lambda \uparrow \sigma \ast \lambda = \tau \cdot \mu \uparrow \mu$, proving the corollary.

Let $A$ and $B$ be alcoves for $\tilde{W}'$ contained in $X^+_{\beta}$. By the theorem we can define $h_{A,B}$ in the obvious way: if $\lambda$ is in the interior of $A$, $\mu$ in the interior of $B$ and $\tilde{W}'$ conjugate to $\lambda$, then $h_{A,B} = h_{\lambda,\mu}$. This is independent of the choice of $\lambda \in A$, $\mu \in B$. Let $C_p$ be the lowest alcove in $X^+_{\beta}$, i.e., $C_p = \{x \in X^+_{\beta}: \langle x + \rho, \alpha_0 \rangle < p\}$, where $\alpha_0$ is the highest short root in $\Delta$. Let $\tilde{W}' \subseteq \{w \in \tilde{W}' : wC_p \subseteq X^+_{\beta}\}$. If $w_1, w_2 \in \tilde{W}'$ define $h_{w_1, w_2} = h_{A,B}$, where $A = w_1 C_p$, $B = w_2 C_p$.

Put a partial order on $\tilde{W}'$ by $w_1 \leq w_2$ iff $h_{w_1, w_2} \neq 0$. It follows from the corollary to the theorem that this is indeed a partial order and that it is
consistent with (i.e., weaker than) the restriction of the Bruhat partial order to \( \tilde{W}^+ \). (See [11].)

There is also a partial order on \( X_\rho \) induced by requiring \( H_{\lambda, \mu} \) to be nonzero. If \( \lambda \) precedes \( \mu \), then there exists \( \beta \in X^+ \) such that \( \lambda + p\beta \uparrow \mu \).

It also follows from the theorem that if \( \lambda \in X_\rho \) and \( \mu \in X^+ \) and if \( H_{\lambda, \mu} \) is nonzero, then \( \lambda \) and \( \mu \) are \( \tilde{W} \)-conjugate, and there exists a \( \beta \in X^+ \) such that \( \lambda + p\beta \uparrow \mu \). Thus if \( \mu \) is in \( X_\rho \), the requirement that \( H_{\lambda, \mu} \) be nonzero induces a partial order on \( X_\rho \), and consequently the matrix \( H \) can be made triangular. Since it has 1's on the diagonal it is therefore invertible over \( \mathbb{Z}[X]^W \). If \( A \) and \( B \) are fundamental domains for \( \tilde{W} \) with \( A \) contained in \( X_\rho \otimes \mathbb{R} \) and \( B \) in \( X^+ \), then we can define \( H_{A, B} \). Let \( D_p \) be a fixed fundamental domain for \( \tilde{W} \) contained in \( C_p \cdot \). If \( w_1, w_2 \) are elements of \( \tilde{W} \) with \( w_1 \cdot D_p = A \subseteq X_\rho \otimes \mathbb{R} \), \( w_2 \cdot D_p = B \subseteq X^+ \), define \( H_{w_1, w_2} = H_{A, B} \).

Let \( T \) be the normal subgroup of \( \tilde{W} \) consisting of all translations by elements of \( \rho X \). The set \( \{ w \in \tilde{W} : w \cdot D_p \subseteq X_\rho \otimes \mathbb{R} \} \) gives a set of coset representatives of \( T \) in \( \tilde{W} \). Since \( \tilde{W}/T \simeq W \), this gives a one-to-one correspondence between this set and \( W \), and so it is possible to define \( H_{\sigma, \tau} \) for \( \sigma, \tau \in W \). It is shown in [3, Lemma 6] that \( H_{\sigma, \tau} \) is equal to the Hulskurkar–Jantzen–Verma element \( \det(\tau x(s, -E_{\sigma, \rho} - \rho)) \). We thus have an induced partial order on \( W \), which is the same as the one described in [4]. However, it is more natural to consider it as a partial order on the subset of \( \tilde{W} \) described above. It then follows that if \( w_1, w_2 \) are in this subset, and if \( w_1 \) precedes \( w_2 \), then \( w_1 \in \tilde{W}^+ \) and there exists \( \beta \in pX^+ \) such that \( w_1 \circ T_\beta \) is in \( \tilde{W}^+ \) and \( w_1 \circ T_\beta \leq w_2 \) in the Bruhat partial order on \( \tilde{W}^+ \), where \( T_\beta \) is a translation by \( \beta \).

The fact that \( H \) can be inverted also follows from [4] for the \( \tilde{W} \)-regular part of \( H \). It was proved in [3] for all of \( H \) using projectives in Jantzen's category of \( u \cdot T \) modules. The argument given here is simpler and more elementary.

We conclude this section by giving another expression for \( h_{\lambda, \mu} \) which will be useful later on. We first note

\textbf{Lemma 3.} For every \( \lambda, \mu \in X^+ \), \( \chi \in \mathbb{Z}[X]^W \), Mult(\( \chi(\lambda), \chi(\mu) \)) = Mult(\( \chi(\mu), \chi(\lambda) \)).

\textbf{Proof.} By linearity it is enough to show this for \( \chi = s(\nu) \) for \( \nu \in X^+ \). Now \( \text{Mult}(\chi(\lambda), s(\nu) \chi(\mu)) = \sum \text{sgn} \tau \), the sum being over all pairs \( (\tau, \nu') \), \( \tau \in W \), \( \nu' \in W\nu \) such that \( \nu' + \mu = \tau(\lambda + \rho) - \rho \). But if this last equation is satisfied then

\[ \tau^{-1}(\mu + \rho) - \rho = -\tau^{-1}\nu' + \lambda = -\tau^{-1}\sigma_0(\nu^*)' + \lambda, \]

(1)

where \( \nu^* = -\sigma_0 \nu \) and \( (\nu^*)' \) is in \( W\nu^* \). But \( \sum \text{det} \tau \), the sum over all pairs...
\((\tau, \tau^{-1} \sigma_0 (v^*)')\) satisfying (1) gives precisely \(\text{Mult}(\chi(\lambda), s(v^*) \chi(\mu)) = \text{Mult}(\chi(\lambda), s(v) \chi(\mu))\).

**Proposition 1.** Let \(\lambda, \mu \in X^+\), where \(\lambda = p\lambda_1 + \lambda_0\), \(\lambda_1 \in X^+\), \(\lambda_0 \in \mathcal{P}_\rho\). Then \(h_{\lambda, \mu} = \text{Mult}(s((p - 1)\rho - \lambda_0) \chi(\lambda_1)^{Fr} \chi(\mu))\).

**Proof.** By definition and Lemma 3,

\[
\begin{align*}
    h_{\lambda, \mu} &= \text{Mult}(\chi(\lambda_1)^{Fr} s((p - 1)\rho - \lambda_0) \chi(\mu)) \\
    &= \text{Mult}(\chi(\mu), s((p - 1)\rho - \lambda_0) \chi(\lambda_1)^{Fr} s(\lambda_1) \chi(\mu)) \\
    &= \text{Mult}(s((p - 1)\rho - \lambda_0) \chi(\lambda_1)^{Fr} \chi(\mu)).
\end{align*}
\]

2. **Calculating the** \(h_{\lambda, \mu}\)

We can approach the question of calculating the \(h_{\lambda, \mu}\) in two ways: either given \(\lambda\) find the values of the nonzero \(h_{\lambda, \mu}\), or given \(\mu\), do the same.

Let us consider the first question. Our first observation is that if \(\lambda\) is far enough inside the dominant Weyl chamber, then so is \(\lambda\) for every \(\sigma \in W\). Hence if \(\sigma \ast \lambda = \tau \cdot \mu\) for \(\mu \in X^+\), then \(\tau\) is the identity. In this case \(h_{\lambda, \mu}\) is nonzero precisely when \(\mu = \sigma \ast \lambda\) for some \(\sigma \in W\), when it is 1. The set of all such \(\mu\) form a set which is symmetric with respect to \(p\lambda_1 + (p - 1)\rho\). This gives a generic pattern for \(\lambda\) far enough inside \(X_\rho^+\).

We claim that all of the \(h_{\lambda, \mu}\) can be described by a finite number of patterns (so that even the \(\lambda\) close to walls of Weyl chambers are included). To see this, let \(M\) be such that if \(\langle \lambda, \alpha_i \rangle \geq M\) for all \(i\), then \(\langle \sigma \ast \lambda, \alpha_i \rangle > 0\) for all \(i\). Let \(S\) be a subset of \(\{1, \ldots, l\}\), and consider those \(\lambda\) close to the walls \(\langle x, \alpha_i \rangle = 0\), \(i \in S\), that is, for which there exists \(\sigma \in W\) such that \(\langle \sigma \ast \lambda, \alpha_i \rangle < 0\) for \(i \in S\). Every \(\lambda_1 \in X^+\) can be written uniquely as \(\lambda_1 = \lambda_1' + \lambda_1''\), where \(\lambda_1', \lambda_1'' \in X^+\) and \(\langle \lambda_1', \alpha_i \rangle = 0\) for \(i \in S\) and \(\langle \lambda_1'', \alpha_i \rangle = 0\) for \(i \in S\). Then for all but a finite number of the \(\lambda \in X^+\) under consideration \(\langle \lambda_1', \alpha_i \rangle \geq M\) for \(i \in S\). If \(\langle \lambda_1'', \alpha_i \rangle \geq M\) for \(i \in S\), is large enough and if \(\tau\) is the element of \(W\) such that \(\tau(\sigma \ast \lambda + \rho) \in X^+\), then \(\tau\) is generated by the reflections \(\sigma_{\alpha_i}, i \in S\). If \(v \in X^+\) is such that \(\langle v, \alpha_i \rangle = 0\) for \(i \in S\), then \(\tau v = v\), so if \(\sigma \ast \lambda = \tau \cdot \mu, \mu \in X^+\), then \(\sigma \ast (\lambda + pv) = \sigma \ast \lambda + pv = \tau \cdot (\mu + pv), \) so \(h_{\lambda + pv, \mu + pv} = h_{\lambda, \mu}\). Thus we have a generic pattern here too.

To illustrate consider the case of \(G_2\) and take \(\alpha_1\) short and \(\alpha_2\) long. Consider those \(\lambda\) for which \(\lambda_0\) is in the bottom alcove \(\mathcal{C}_\rho\). The nongeneric degenerate cases occur for eight values of \(\lambda_1 = (a, b)\), where \(0 \leq a \leq 3\) and \(b = 0\) or \(1\). In the remaining cases \(\sigma \ast \lambda\) for \(\sigma \in W\) is dominant with the following exceptions: If \(\lambda_1 = (0, n)\) with \(n > 1\) and \(\sigma\) is either \(\sigma_{\alpha_2}, \sigma_{\alpha_1} \sigma_{\alpha_2}\),
2. Labellings of alcoves for rank 2 types.

If $\sigma_{\alpha_2} \sigma_{\alpha_1} \sigma_{\alpha_2}$, or $(\sigma_{\alpha_1} \sigma_{\alpha_2})^2$, then $\mu = \sigma_{\alpha_1} \cdot (\nu \ast \lambda)$ is in $X^+$ and $h_{\lambda, \mu} = -1$. If $\lambda_1 = (n, 0)$ or $(n, 1)$ with $n > 3$ and $\sigma$ is either $\sigma_{\alpha_1}$, $\sigma_{\alpha_2}$, $\sigma_{\alpha_1} \sigma_{\alpha_2} \sigma_{\alpha_1}$, or $(\sigma_{\alpha_1} \sigma_{\alpha_2})^2$, then $\mu = \sigma_{\alpha_2} \cdot (\sigma \ast \lambda)$ is in $X^+$ and $h_{\lambda, \mu} = -1$. In the remaining cases for $\lambda_1$, $h_{\lambda, \mu}$ follows the generic pattern.

Now consider the question of given $\mu$, for which $\lambda$ will $h_{\lambda, \mu}$ be nonzero. Again, if $\mu$ is far enough inside $X^+$, then there will be a generic pattern and

Fig. 1. Labellings of alcoves for rank 2 types.

Again, if $\mu$ is far enough inside $X^+$, then there will be a generic pattern and

Fig. 2. Configuration of special points associated with an alcove of lowest type for $A_2$ and $B_3$. 
the desired $\lambda$ will be those for which there exists $\sigma \in W$ such that $\sigma \ast \lambda = \mu$. Let $v$ be a special point (an element of $X$ of the form $pv - \rho$, $v \in X$), and let $\Pi_v$ be the box $X_p + v - (p - 1)\rho$. If $C$ is an alcove in $\sigma \ast \Pi_v$ (star action with respect to $v$), let $C'$ be the alcove in $X_p$ with $\sigma \ast (C' + v - (p - 1)\rho) = C$, and label $v$ with $C'$. For each $C$ we obtain a configuration of special points labeled this way, and if $w \in \widetilde{W}$, then the configuration for $w \cdot C$ is obtained from that of $C$ by applying $w$ to it. The alcoves $A$ for which $h_{\lambda, C}$ is nonzero are those for which $A = C' + v - (p - 1)\rho$, where $v$ is a special point, $C'$ is an alcove in $X_p$ and $v$ is labelled by $C'$ in the configuration for $C$.

The configurations for $C$ when $C$ is a translate of $C_p$, in types $A_2, B_2$ are shown in Fig. 2. The alcoves of $X_p$ themselves are denoted by integers as shown in Fig. 1.

It is possible to give generic degenerate patterns here too.

Next consider the question of computing $H$. Here we wish to see when we can have $\sigma \ast \lambda = \tau \cdot \mu$ for $\mu \in X_\rho$. When this happens $H_{\lambda_0, \mu} = \det \tau \chi(\lambda_1)$. In

![Diagram](https://via.placeholder.com/150)

**Fig. 3.** Computation of $H$ for $G_2$. 
the rank 2 cases the calculations can be done diagramatically, as illustrated in Fig. 3 for $G_2$.

The nonzero $H_{A,B}$ are given by intersections of images of alcoves in $X_p$ under the dot action with those under a star action with respect to a special point. These intersections are pictured in the shaded areas. We will describe the parts of the matrices corresponding to $\lambda$ and $\mu$ in the interior of an alcove in the rank 2 cases. For $A_2$ and $B_2$ when $\mu \in X_p$, $H_{\lambda,\mu} = h_{\lambda,\mu}$ so if $H_{\lambda,\mu}$ is nonzero $\lambda$ and $\mu$ are $\tilde{W}$-conjugate. For $G_2$, $X' = X$ so this is also true in this case. Therefore we can speak of $H_{A,B}$, where $A$ and $B$ are alcoves. If $\lambda$ or $\mu$ is on the boundary of an alcove we get the same values as for the interior except that we disregard those for which the corresponding $\lambda$ or $\mu$ is not in $X^+$.

For $A_2$, $H$ is the identity matrix. For $B_2$, $(H_{A,B})$ is

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 0 & -1 & 0 \\
2 & & 1 & 0 & -1 \\
3 & & & 1 & 0 \\
4 & & & & 1 \\
\end{array}
\]

For $G_2$ it is

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
G_2: 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -Y & 0 & 0 \\
2 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -Y & 0 \\
4 & & & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
5 & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & & & & & & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
8 & & & & & & & 1 & 0 & -1 & 0 & 0 & 0 \\
9 & & & & & & & & 1 & 0 & -1 & 0 & 0 \\
10 & & & & & & & & & 1 & 0 & 0 & 0 \\
11 & & & & & & & & & & 1 & 0 & 0 \\
12 & & & & & & & & & & & 1 & 0 \\
\end{array}
\]

Here $Y = X(1, 0)$.

It may be noticed that in each of these matrices there are zeros just above the diagonal. In fact if $\lambda, \mu$ are in $X_p$ and $\mu$ lies in an alcove just above the one in which $\lambda$ lies, and if $\mu$ is obtained from $\lambda$ by a reflection in the wall
separating these two alcoves, then both $H_{\lambda,\mu}$ and $h_{\lambda,\mu}$ are 0. This follows from Proposition 2 which shows a bit more than this.

**Proposition 2.** Suppose $\mu$ and $\lambda$ are $\tilde{W}'$-regular and are $\tilde{W}'$-conjugate by a reflection in a hyperplane of the form $\langle x + \rho, \alpha^* \rangle = kp$, $0 < k < \langle \lambda_1 + \rho, \alpha^* \rangle$. Then $h_{\lambda,\mu} = 0$.

**Proof.** Suppose the reflection is in the hyperplane $\langle x + \rho, \alpha^* \rangle = kp$, so that $\mu = \sigma_\alpha \cdot \lambda + k \rho$. Suppose also that $h_{\lambda,\mu} \neq 0$. Then we can find $\sigma, \tau \in W$ such that

$$\mu = \sigma \cdot \lambda + p(\tau(\rho + \lambda_1) - \sigma(\rho + \lambda_1)).$$

We then must have $\sigma = \sigma_\alpha$ and $k \alpha = \tau(\rho + \lambda_1) - \sigma_\alpha(\rho + \lambda_1) = \tau(\rho + \lambda_1) - (\rho + \lambda_1) + \langle \rho + \lambda_1, \alpha \rangle \alpha$, or

$$-(\langle \rho + \lambda_1, \alpha^* \rangle - k) \alpha = \tau(\rho + \lambda_1) - (\rho + \lambda_1). \tag{2}$$

Thus $-(\langle \rho + \lambda_1, \alpha^* \rangle - k) \tau^{-1} \alpha = \rho + \lambda_1 - \tau^{-1}(\rho + \lambda_1) \in \Delta^+$ so $\tau^{-1} \alpha \in \Delta^-$. Now apply $\sigma_\alpha$ to both sides of (2), to get

$$\langle \rho + \lambda_1, \alpha^* \rangle - k) \alpha = \sigma_\alpha \tau(\rho + \lambda_1) - (\rho + \lambda_1) + \langle \rho + \lambda_1, \alpha^* \rangle \alpha$$

or $-k \alpha = \sigma_\alpha \tau(\rho + \lambda_1) - (\rho + \lambda_1)$. Thus $k \tau^{-1} \alpha = \rho + \lambda_1 - \tau^{-1} \sigma_\alpha(\rho + \lambda_1)$, so $\tau^{-1} \alpha \in \Delta^+$, a contradiction. This proves the proposition.

Finally, we note a simple symmetry property of $H$.

**Lemma 4.** Suppose $\lambda, \mu \in X_p$ are $\tilde{W}$-conjugate with $\mu \equiv w \cdot \lambda (\text{mod } pX)$ $w \in W$. Let $\lambda' = (p - 2)\rho - \lambda$, $\mu' = (p - 2)\rho - \mu$. Then $H_{\lambda,\mu} = \det w H_{\mu',\lambda'}$.

**Proof.** Suppose $\beta \in X^+$, $\sigma, \tau \in W$, and $\sigma(\mu + p\beta) = \tau \cdot \mu$. Then $\tau \cdot \lambda' = \sigma(\mu' + p\beta)$, so $\det \tau H_{\lambda,\mu} = \det \sigma H_{\mu',\lambda'}$. Since $w - \tau^{-1} \sigma$, we are done.

3. **Applications**

We discuss some applications of the $h_{\lambda,\mu}$. Let $\chi_\rho(\lambda)$ be the formal character of the irreducible $G$-module with highest weight $\lambda$. Consider the following two $\mathbb{Z}$ bases of $\mathbb{Z}[X]^+$:

$$S_1 = \{\chi_\rho(\lambda) : \lambda \in X^+\}, \quad S_2 = \{\chi(\lambda_1)^{\text{Fr}} \chi_\rho(\lambda_0) : \lambda_0 \in X_p, \lambda_1 \in X^+\}.$$
For $\lambda, \mu \in X^+$, define
\[
b_{\mu, \lambda} = \text{Mult}_S(\chi_p(\lambda), \chi(\mu)), \quad \tilde{b}_{\mu, \lambda} = \text{Mult}_S(\chi(\lambda_1)^T, \chi_p(\lambda_{0}), \chi(\mu)).
\]

Even when $\lambda \in X_p$ it is not necessarily true that $\tilde{b}_{\mu, \lambda}$ is the same as $b_{\mu, \lambda}$. In fact we will see that for given $\lambda$, $\tilde{b}_{\mu, \lambda}$ is nonzero for only finitely many $\mu$, while there are infinitely many $\mu$ for which $b_{\mu, \lambda}$ is nonzero. However, Jantzen shows in [9] that if $\lambda \in X_p$ and $\tilde{b}_{\mu, \lambda}$ is nonzero, then $\tilde{b}_{\mu, \lambda} = b_{\mu, \lambda}$, provided $p$ is large enough.

For $\lambda \in X_p$, $\mu \in X^+$, define $B_{\mu, \lambda} = \text{Mult}_p(\chi_p(\lambda), \chi(\mu))$. Here of course we are taking $S = \{\chi_p(\lambda) : \lambda \in X_p\}$. We have $b_{\mu, \lambda} = \text{Mult}(\chi(\lambda_1), B_{\mu, \lambda})$.

Let $B$ be the matrix $(B_{\mu, \lambda})$, $\mu, \lambda \in X_p$, $B$ and $b$ the matrices $(b_{\mu, \lambda})$, $\mu, \lambda \in X^+$. It is known that if $b_{\mu, \lambda}$ is nonzero, then $\lambda \uparrow \mu$ (see [1]). Since $\tilde{b}_{\mu, \lambda}$ being nonzero implies $b_{\mu, \lambda}$ is also nonzero, the same is true for $\tilde{b}_{\mu, \lambda}$. Thus the matrices $b$, $B^T$, and $H$ can be simultaneously triangulated, and the same is true for $B^T$ and $H$. Let $g = h^{-1}$, $a = b^{-1}$, $\tilde{a} = \tilde{b}^{-1}$, $G = H^{-1}$, $A = B^{-1}$. Denote the $\lambda - \mu$ entry of $g$ by $g_{\lambda, \mu}$, etc.

Our first application is to derive generic patterns for the $\tilde{b}_{\mu, \lambda}$. These have been derived by Jantzen in [8], but our description will be a bit more explicit. From the definitions it follows easily that
\[
B_{\xi, \lambda} = \sum_{\mu, \nu \in X_p} B_{\mu, \lambda} G_{\mu, \nu} H_{\nu, \xi} \quad \text{for} \quad \xi \in X^+, \lambda \in X_p.
\]
(See [3]). It follows that for $\xi, \lambda \in X^+$,
\[
\tilde{b}_{\xi, \lambda} = \text{Mult} \left( \chi(\lambda_1), \sum_{\mu, \nu \in X_p} B_{\mu, \lambda_0} G_{\mu, \nu} H_{\nu, \xi} \right).
\]

Let $C_{\lambda_0, \tau} = \sum_{\mu \in X_p} R_{\mu, \lambda_0} G_{\mu, \tau}$. Then by Lemma 1 it follows that
\[
0_{\xi, \lambda} = \sum_{\nu \in X_p} \text{sgn} \tau \text{Mult}(\eta, C_{\lambda_0, \tau}) h_{\nu + p\beta, \xi}.
\]
Hence $b_{\xi, \lambda}$ is nonzero precisely when there exist $\sigma, \tau, \tau_1 \in W$ and $\nu \in X_p$, $\beta \in X^+$, $\eta \in X$ such that $\eta$ is a weight of $C_{\lambda_0, \nu}$ and $\tau \cdot \xi = \sigma \ast (\nu + p(\beta))$ and $\tau_1 \cdot \beta = \lambda_1 - \eta$. The multiplicity is then $\sum \det \tau \text{Mult}(\eta, C_{\lambda_0, \tau})$, the sum over all such $\nu$ and $\eta$.

In case $\lambda$ and $\xi$ are sufficiently deep in the interior of the fundamental Weyl chamber we have $\tau = \tau_1 = \text{identity}$ and obtain a generic pattern. This can be interpreted for either $\lambda$ or $\xi$ fixed. Suppose $\lambda$ is fixed and we ask for
the $\xi$ for which $\tilde{b}_{t,\lambda}$ is nonzero. Since $C_{\lambda_0,\nu} \in \mathbb{Z}[X]^{W}$, $\text{Mult}(\sigma \eta, C_{\lambda_0,\nu}) = \text{Mult}(\eta, C_{\lambda_0,\nu})$ for any $\sigma \in W$. Hence these $\xi$ will form a symmetric pattern about the special point $v = p(\lambda_1 + \rho) - \rho$. Therefore it is enough to give the values of $\tilde{b}_{t,\lambda}$ for $\xi$ in $\mathcal{G}_v$. This corresponds to taking $\xi = p(\lambda_1 - \eta) + v$ with $v \in X_\nu$ and $\eta$ a weight of $C_{\lambda_0,\nu}$ which is anti-dominant. We then have $\tilde{b}_{t,\lambda} = \text{Mult}(\eta, C_{\lambda_0,\nu})$.

**Example.** Consider the case of $G_2$. Again we only describe the pattern for $\lambda$ and $\mu$ $\tilde{W}$-regular. We have to know the matrix $B^T H^{-1}$. Inverting the matrix $H$ obtained in Section 2, we find the following form for $H^{-1}$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>Y</td>
<td>-1</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

From [7, p. 140] we can deduce the matrix $B^T$. It is

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Y</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Hence we have for $B^TH^{-1}$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$Y - 1$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$Y + 2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$Y + 2$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The weights of $Y = \chi(1, 0)$ are $(1, 0)$, $(-1, 1)$, $(2, -1)$, $(-2, 1)$, $(1, -1)$, $(-1, 0)$, and $(0, 0)$. If $\nu \in X_\rho^-$, the weights $\eta$ of $\chi(1, 0)$ such that

![Diagram](image-url)

**Fig. 4.** The nonzero $b_{\lambda, \mu}$ for $\lambda$ fixed in an alcove of lowest type and for which $\xi \in \mathcal{H}_\rho^-$, in case $G_2$. 


\[ \xi = p(\lambda_1 - \eta) + \nu \] is in \( \mathcal{C}_v^- \) are \((-1,0)\) and \((0,0)\). To illustrate, consider the case when \( \lambda_0 \) is in \( C_p \). Then the values of \( b_{L,\lambda} \) for \( \xi \in \mathcal{C}_v^- \) are read off from the first row of \( B^T H^{-1} \) and are shown in Fig. 4, where \( n \times' \)'s in an alcove mean that \( b_{L,\lambda} = n \) for \( \xi \) the weight which is \( \tilde{W}' \)-conjugate to \( \lambda \) in that alcove.

Now suppose that \( \xi \) is fixed and we ask for the \( \lambda \) which make \( b_{L,\lambda} \) nonzero. To describe the pattern we generalize the labelling of the configuration of special points associated with \( \xi \) given in Section 2. Thus to the special point \( \nu \) we associate a set of pairs of the form \((C, \eta)\), where \( C \) is an alcove and \( \eta \in X \), and where specifically \((C, \eta)\) belongs to this set if \( \nu - p\eta \) was labelled by \( C \) in the old labelling. Now fill in the box \( \mathcal{C}_v^- \) by putting \( n \times' \)'s in the alcove containing \( \nu - p\eta + \nu \) if \( \eta \) is a weight of \( C_{\lambda_0, v} \) of multiplicity \( n \), where \( \nu \) contains the label \((C, \eta)\) and \( \lambda_0 \in C \). Note that this depends only on the labelling of \( \nu \) and is independent of the actual point \( \nu \) or \( \xi \). The resulting pattern gives the values of the \( \delta_{L,\lambda} \).

**Example.** Consider the case of \( G_2 \) again. The configuration of special points associated with \( \xi \) when \( \xi_0 \) is in \( C_p \), together with their labelling, is

![Diagram](image-url)
shown in Fig. 5. The symbols (a)–(p) are explained below. Every other (generic) configuration is obtained by applying the appropriate element of $\tilde{W}$.

The filling in of the boxes $\Theta_{\eta}$ is the same as that given by Jantzen [8, pp. 457–458]. To complete the picture we only have to specify the labels for each of the diagrams. We only give the labels which are relevant, i.e., we only display those $\eta$ which are weights of $C_{\lambda, \nu}$. We also abbreviate $(C, 0)$ by $C$ if $\eta = 0$. The labellings are then in order.

(a) \{12, (11, (−1, 0)), (10, (−2, 1)), (9, (1, −1))\},
(b) \{11, (12, (1, 0)), (10, (−1, 1)), (9, (2, −1))\},
(c) \{10, (12, (2, −1)), (11, (1, −1))\},
(d) \{9, (12, (−1, 1)), (11, (−2, 1))\},
(e) \{8, (12, (1, −1)), (10, (−1, 0))\},
(f) \{7, (12, (−2, 1)), (9, (−1, 0))\},
(g) \{6, (11, (2, −1)), (10, (1, 0))\},
(h) \{5, (11, (−1, 1)), (9, (1, 0))\}.

Jantzen indicates that the next pattern occurs twice. This corresponds to the following two labellings:

(i) \{4, (10, (1, −1))\},
(j) \{(12(−1, 0))\};

the next two patterns:

(k) \{3, (9, (−2, 1))\},
(l) \{2\};

the next pattern which occurs four times corresponds to:

(m) \{1\},
(n) \{(11, (1, 0))\},
(o) \{(10, (2, −1))\},
(p) \{(9, (−1, 1))\}.

Notice how easily Jantzen's diagrams can be computed from $B^T H^{-1}$.

In [10] Lusztig states some conjectures as to the values of the $b_{\mu, \lambda}$. The patterns he describes are for the $\mu$ which make $b_{\mu, \lambda}$ nonzero for fixed $\lambda$, and hold for $\lambda$ far enough away from the walls of $X_\mu^+$ in the lowest “$p^2$ alcove.” However, if we consider $\tilde{b}_{\mu, \lambda}$ instead of $b_{\mu, \lambda}$, then the patterns (conjecturally) hold for all $\lambda$, as long as $\lambda$ is far enough away from a wall.

The method given above for the calculation of the generic pattern depends on knowing the matrix $B$, i.e., knowing the (nongeneric) pattern for $X_\mu$. However, the verification of Lusztig’s conjectures would give a method of
calculating the generic patterns directly, so it is interesting to note that we
can turn the procedure around and calculate the pattern for \(X_\mu\), and indeed
any degenerate pattern, from the generic pattern. The possibility of doing this
is noted by Humphreys [6], but his description is valid only for the special
case he considers. We will give here the correct general algorithm.

Consider again formula (3). We want to see how the picture compares to
the generic one when in the equation \(\tau \cdot \bar{\zeta} = \sigma \ast (v + p(\lambda_1 - \eta))\) we have \(\tau\) not
the identity. Consider first the situation when \(\lambda\) is taken to be fixed. In this
case the description is very simple and is as follows: In the generic picture
for \(\lambda\), consider all those \(\theta\) which are not dominant, and bring them into the
dominant region by applying \(\tau^{-1}, \tau \in W\). Every time this is done we get a
contribution to \(b_{\xi,\lambda}, \zeta = \tau^{-1} \cdot \theta\), of \(\det \tau\).

Now consider the case when \(\xi\) is taken to be fixed. The description of how
the special configuration is obtained from the general configuration is as
follows: In the general configuration, consider a special point \(v\) with its
generalized labelling. The only special points which contribute are those
which are in the interior of a Weyl chamber (with respect to origin \(-\rho\)) and
which have as part of their labelling \((C, \eta)\), where the point \((C, 0)\) is also in
the interior of a Weyl chamber, and where \(\eta\) is a weight of \(C_{\lambda,\nu}\) as above.
Suppose \(v\) is such a special point. Fill in the box \(\mathcal{B}_v\) as before, but using
only those parts of the label \((C, \eta)\) for which the point with label \((C, 0)\) is in
the interior of a Weyl chamber. Now let \(\zeta\) be the unique element of \(W\) such
that \(\zeta \cdot v\) is in \(X^+\), and shift this pattern to \(\zeta \cdot v\) by the translation \(v \rightarrow \zeta \cdot v\),
with a sign adjustment of \(\text{sgn} \zeta\).

To see why this is so, note that the conditions \(\tau \cdot \bar{\zeta} = \sigma \ast (v + p\beta)\) and
\(\tau_1 \cdot \beta = \lambda_1 - \eta\), are equivalent to \(\zeta = \tau^{-1} \sigma \ast (v + p(\tau^{-1} \cdot \beta))\) and
\(\tau^{-1} \cdot \beta = (\tau_1)^{-1} \cdot \lambda_1 - (\tau_1 \tau)^{-1} \eta\). Thus each time we have a contribution of \(\text{sgn} \tau\tau_1\) to
\(b_{\xi,\lambda}\), in the general configuration we would put an \(x\) in the alcove containing
\(v + p(\tau^{-1} \cdot \beta + (\tau_1 \tau)^{-1} \eta)\). Since the multiplicity of \(\eta\) as a weight of \(C_{\lambda,\nu}\) is
the same as that of \((\tau_1 \tau)^{-1} \eta\), and since the element \(\zeta\) of \(W\) such that
\(\zeta \cdot (\tau_1 \tau)^{-1} \cdot (\tau_1 \tau)^{-1} \eta\) is dominant is \(\zeta = \tau_1 \tau\), we see that our description is
correct.

As an example, consider the case of \(G_2\) and \(\xi\) in the alcove marked with \(\otimes\)
in Fig. 6. These are 7 relevant special points, 4 lying inside \(X^+\) and 3
outside. These are labelled (a)–(g) in Fig. 6, where these labels have the
following meanings:

(a): \{\((12, 0), (10, (-1, 0)), (9, (-1, 1))\}\,
(b): \{\((10, 0), (12, (1, 0))\}\,
(c): \{\((6, 0), (10(2, -1))\}\,
(d): \{\((10, (1, 0))\}\,
(e): \{\((9, 0), (12, (1, -1))\}\,
(f): \{(5, 0), (9, (2, -1))\},
(g): \{(2, 0)\}.

The configurations associated with (e) and (g) (which are read off from the columns of $B^T H^{-1}$) are translated to (a) with a minus sign to partially cancel some of the configuration there. Similarly, the configuration for (f) is brought in to (b). The resulting pattern is as shown.
The existence of generic decomposition patterns was proved by Jantzen [8]. Our methods are simpler than his due to the fact that we have a simpler form for the Hulsurkar elements. Jantzen [9] gives another, simpler proof of the existence of the generic patterns, by considering his category of \( u_n - T \) modules. However, his methods there are not suitable for the actual computation of the patterns. The main difference between his methods there and ours is that he goes outside to the ring \( \mathbb{Z}[X] \), while we remain throughout in the ring \( \mathbb{Z}[X]^W \).

For our next application we will obtain a formula for the characters of certain projective modules. For \( \lambda, \mu \in X^+ \), define \( \psi_\lambda = st \cdot \psi_\lambda \), where \( \psi_\lambda = \sum_{\mu, \nu} b_{\mu, \nu} g_{\mu, \nu} \chi(v_1) s((p-1)\rho - v_0) \), where \( (g_{\mu, \nu}) \) is the inverse matrix to \( (h_{\mu, \nu}) \), \( \mu, \nu \in X^+ \). It follows from formula (3) and the fact that for fixed \( \mu \) (resp. \( \nu \)), \( h_{\mu, \nu} \) is nonzero for only finitely many \( \nu \) (resp. \( \mu \)), that the sum defining \( \psi_\lambda \) is in fact finite. Since the matrix \( B^T \Sigma \) is triangular with 1's on the diagonal the set \( \{\psi_\lambda; \lambda \in X^+\} \) forms a \( \mathbb{Z} \) basis for \( \mathbb{Z}[X]^W \).

**Lemma 5.** For every \( \lambda, \mu \in X^+ \), \( \text{Mult}(st, \psi_\lambda \chi(\mu))^F \chi(\mu) = \delta_{\lambda, \mu} \). Moreover this property characterizes the set \( \{\psi_\lambda; \lambda \in X^+\} \).

**Proof.** The left-hand side is equal to

\[
\sum_{\theta, \tau \in X^+} \text{Mult}(st, \tilde{\delta}_{\theta, \lambda} g_{\theta, \tau} \chi(v_1) \chi(\mu))^F s((p-1)\rho - v_0) \chi(\mu) \chi(\eta))
\]

\[
= \sum_{\theta, \tau, \eta \in X^+} \tilde{\delta}_{\theta, \lambda} g_{\theta, \tau} \tilde{\delta}_{\mu, \nu} \text{Mult}(st, \chi(v_1) \chi(\eta))
\]

\[
= \sum_{\theta, \tau, \eta \in X^+} b_{\theta, \lambda} g_{\theta, \tau} a_{\mu, \eta} h_{\tau, \eta} \quad \text{(by Proposition 1)}
\]

\[
= \sum_{\theta, \tau} \tilde{\delta}_{\theta, \lambda} \tilde{\delta}_{\mu, \nu} = \delta_{\lambda, \mu}.
\]

The second statement is immediate because \( \{\psi_\lambda; \lambda \in X^+\} \) is a \( \mathbb{Z} \) basis for \( \mathbb{Z}[X]^W \).

Suppose \( \lambda, \mu \in X^p \). Then if \( \alpha \in X^+ \), \( \text{Mult}(\chi(\alpha)^F st, \psi_\lambda \chi(\mu)) = \text{Mult}(st, \chi(\alpha)^F \psi_\lambda \chi(\mu)) \), which by Lemma 5 is 0 unless \( \alpha = 0 \) and \( \lambda = \mu \). Hence \( \text{Mult}(st, \psi_\lambda \chi(\mu)) = \delta_{\lambda, \mu} \). This shows that for \( \lambda \in X^p \), \( \psi_\lambda \) coincides with the character \( \Psi_\lambda \) defined in [3]. (This is the character of the projective indecomposable \( u - T \) module parametrized by \( \lambda \)).

Lemma 5 suggests that there is some category whose projective indecomposables are given precisely by \( \{\psi_\lambda; \lambda \in X^+\} \) (cf. [3]). Presumably this would be the category of \( G \) modules which have a filtration with successive factors of the form \( V^F \otimes L \), where \( V \) is a Weyl module and \( L \) is an
irreducible $G$ module, with highest weight in $X_p$, and with morphisms defined appropriately.

**Lemma 6.** For $\lambda \in X^+$, $\psi_\lambda = \chi(\lambda_1)^{Fr} \psi_{\lambda_0}$.

**Proof.** We have

$$\text{Mult}(st, \chi(\lambda_1)^{Fr} \chi_0, \chi_{\lambda_1}) = \text{Mult}(\chi(0), \chi(\lambda_1) \chi(\mu_0)) \delta_{\lambda_0, \mu_0}$$

$$= \delta_{\lambda_1, \mu_1} \delta_{\lambda_0, \mu_0} = \delta_{\lambda, \mu}.$$

By Lemma 5 we must then have $\psi_\lambda = \chi(\lambda_1)^{Fr} \psi_{\lambda_0}$.

The following result was originally proved by Jantzen [9, Sect. 5.9], using his theory of $u - T$ modules.

**Proposition 3.** For $\lambda, \mu \in X^+$,

$$\text{Mult}(\chi(\mu), \Psi_\lambda) = \text{Mult}(\chi(\lambda_1)^{Fr} \chi_{\lambda_0}, \chi(\mu)) = \delta_{\mu, \lambda}.$$

**Proof.**

$$\text{Mult}(\chi(\mu), \Psi_\lambda) = \text{Mult}(\chi(\mu), \text{st} \cdot \psi_\lambda)$$

$$= \text{Mult}(\text{st}, \psi_\lambda \chi(\mu)) = \text{Mult}(\chi_0, \psi_\lambda \chi(\mu))$$

$$= \sum_{\nu \in X^+} \text{Mult}(\text{st}, \psi_\lambda \tilde{\delta}_{\lambda, \nu} \chi_0)$$

$$= \sum_{\nu \in X^+} \delta_{\mu, \nu} \delta_{\lambda, \nu} = \delta_{\mu, \lambda},$$

as required.

We have seen that if $\lambda$ and $\mu$ are related by a reflection in a wall separating the two alcoves which they are in, then $H_{\lambda, \mu} = 0$. Consequently, $G_{\lambda, \mu} = 0$ in this case also. It follows from [7, Satz 10] that $B_{\mu, \lambda} = 1$ in this case if $\lambda \uparrow \mu$. It follows that in this case, $\sum_{\nu \in X_p} B_{\nu, \lambda} G_{\nu, \mu} = 1$.

Thus in view of the formula for $\psi_\lambda$, $\lambda \in X_p$, given in [3], it follows that we will have $\Psi_\lambda = s((p - 1)\rho - \lambda) \text{st}$ only in the case that $\lambda$ is in the closure of the top alcove of $X_p$. Ballard [2] showed that a certain alternating sum of induced characters constructed by Srinivasan for the finite group has character $s((p - 1)\rho - \lambda) \text{st}$, considered as a Brauer character. He concludes that in certain cases this alternating sum is the character of a projective indecomposable module for the finite group. In view of the connection derived in [3] between the $\Psi_\lambda$ and the projective characters, and from what we have said above, it follows that this will be true only in the case that $\lambda$ is in the closure of the top alcove of $X_p$. 
REFERENCES