$\delta^2$ Transform and Fourier Series of Functions with Multiple Jumps

Emily Jennings

Georgia Institute of Technology

Nebraska Conference for Undergraduate Women in Mathematics, 2012

Work performed at Kansas State University’s REU with students Daniel Muñiz (University of Florida) and Ashley Toth (Rollins College) under Dr. Charles Moore.
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- Heat equation and Joseph Fourier
- Similarly used to find eigensolutions/simple solutions to PDEs.
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Fourier Series

For a function $f$ integrable on $[-\pi, \pi]$, we define the Fourier coefficients by

\[
a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx
\]

\[
a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx
\]

\[
b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx
\]

for each integer $n \geq 1$. 
The $N^{th}$ partial sum of the Fourier series is

$$S_N f(x) := a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx,$$

where $N$ is a positive integer and $x \in [-\pi, \pi]$.

Complex exponentials: Euler’s formula

$$e^{inx} = \cos nx + i \sin nx$$
Fourier Series

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- Complex exponentials: Euler’s formula

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Fourier Series

- Using complex exponentials, we define the Fourier coefficients for a function $f$ integrable on $[-\pi, \pi]$ by,

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx$$

for each integer $n$.

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where $N$ is a positive integer and $x \in [-\pi, \pi]$. 
The Fourier series for any function in $L^2$ converges.

Problem: Convergence of Fourier series of certain functions can be relatively slow.

Example: Functions with jump discontinuities.

If $(s_n) \rightarrow s$, we say that a transformation $t_n$ accelerates the convergence of $(s_n)$ if there exists a $k > 0$ such that each $t_n$ depends only on $s_1$, $s_2$, ..., $s_{n+k}$ and $(t_n)$ converges to $s$ faster than $(s_n)$. 
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The $\delta^2$ Transformation

- What happens when we apply the $\delta^2$ transform, a convergence acceleration method, to the partial sums of the Fourier series?

- For the sequence $(S_N f(x))$, the $\delta^2$ transform results in a sequence with terms as follows:

$$t_N := S_N - \frac{(S_{N+1} - S_N)(S_N - S_{N-1})}{(S_{N+1} - S_N) - (S_N - S_{N-1})},$$

where we set $t_N = S_N$ if the denominator of the fraction is zero.
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Some General Previous Results

- If a complex series and its $\delta^2$ transform converge, then their sums are equal (Tucker, 1967).
- Ciszweski, Gregory, Moore and West (2011) found continuous functions for which the transformed Fourier series fails to converge.
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Throughout this presentation, $f$ is a piecewise smooth function integrable on $[-\pi, \pi]$ having a finite number of jump discontinuities at $a_1, a_2, ..., a_m \in (-\pi, \pi)$.

Let $d_j = f(a_j^+) - f(a_j^-)$ and $d_j^* = f'(a_j^-) - f'(a_j^+)$ for all $j \in \{1, ..., m\}$. (Assume the derivatives $f'(a_j^-)$ and $f'(a_j^+)$ exist and are finite.)
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After some computation, the $N^{th}$ partial sum of the Fourier series is

$$S_N f(x) = \hat{f}(0) + \sum_{k=1}^{N} \left[ \frac{1}{k\pi} \sum_{j=1}^{m} \left[ d_j \sin k(x - a_j) \right] + \epsilon_k \right],$$

where

$$\epsilon_k = \frac{1}{k^2\pi} \sum_{j=1}^{m} \left[ d_j^* \cos k(x - a_j) \right] - \frac{\hat{f}''(k)e^{ikx} + \hat{f}''(-k)e^{-ikx}}{k^2}.$$
Partial Sum of Fourier Series

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We note that $\epsilon_k \in O\left(\frac{1}{k^2}\right)$ and call this the error term.

Applying the $\delta^2$ process and simplifying gives a sequence of transformed partial sums

$$t_N f(x) \approx S_N f(x) - \frac{\left( \sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] \right) \left( \sum_{j=1}^{m} [d_j \sin N(x - a_j)] \right)}{N\pi \sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] - (N + 1)\pi \sum_{j=1}^{m} [d_j \sin N(x - a_j)]}. \quad (1)$$
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Transformed Series

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$$\frac{\left(\sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] \right) \left(\sum_{j=1}^{m} [d_j \sin N(x - a_j)] \right)}{N \pi \sum_{j=1}^{m} [d_j \sin (N + 1)(x - a_j)] - (N + 1) \pi \sum_{j=1}^{m} [d_j \sin N(x - a_j)].}$$

(1)
Two Lemmas

- **Lemma 1: Denominator is bounded above (small).** For a fixed real number $x_0$, the function
  \[ g(x) = \sum_{j=1}^{m} d_j \left[ \sin(N + 1)(x - a_j) - \sin N(x - a_j) \right] \]
  is bounded above by \( \frac{2C}{N} \), where \( C \) is some constant, on a subinterval of \( [x_0, x_0 + \frac{\pi}{N+1}] \).

- **Lemma 2: Numerator is bounded below (not zero).** If the points of discontinuity of \( f \) are restricted to rational multiples of \( \pi \), then the numerator of fraction (1) is bounded below by a positive constant at the points where the denominator is close to zero.
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- **Lemma 2**: Numerator is bounded below (not zero). If the points of discontinuity of $f$ are restricted to rational multiples of $\pi$, then the numerator of fraction (1) is bounded below by a positive constant at the points where the denominator is close to zero.
**Main Theorem**

- **Theorem:** \((t_n)\) does not converge to \(f\). If \(f\) fits our constraints described and if there is at least one value of \(N\) for which the numerator of (1) is not 0, then for each \(N \in \mathbb{N}\), there exist intervals for which the fraction in (1) is greater than or equal to \(M \in \mathbb{R}\).
Rearranging the numerator of (1), define

\[ r_N := \sum_{j=1}^{m} \left[ \cos N a_j + i \sin N a_j \right] = \sum_{j=1}^{m} d_j e^{i N a_j}. \]

We can show that if for any \( N \), \( r_N = 0 \) or \( r_{N+1} = 0 \), then the numerator of (1) is 0 for all \( N \in \mathbb{N} \). (The theorem does not apply.)
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We can show that if for any \( N \), \( r_N = 0 \) or \( r_{N+1} = 0 \), then the numerator of (1) is 0 for all \( N \in \mathbb{N} \). (The theorem does not apply.)
Example 1. Consider a function $f$ with jump discontinuities at $-\frac{3\pi}{4}$, $-\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{3\pi}{4}$ such that $d_1 = d_3$ and $d_2 = d_4$.

Then $r_N = 0$ for $N$ odd, so the numerator of (1) is 0 for all $N$. 
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Example 1

Figure: $\delta^2$ applied to partial sums of a function with **equal jumps** ($d_1 = d_3 = -2$, $d_2 = d_4 = 1$).

Figure: $\delta^2$ applied to partial sums of a function with **unequal jumps** ($d_1 = d_3 = -2$, $d_2 = 1$, $d_4 = 2$).
Example 2. Consider a function $f$ with jump discontinuities at $-\frac{2\pi}{3}$, 0, and $\frac{2\pi}{3}$ such that $d_1 = d_2 = d_3$.

Then $r_N = 0$ for $N$ not a multiple of 3, so the numerator of (1) is 0 for all $N$. 
Example 2. Consider a function $f$ with jump discontinuities at $-\frac{2\pi}{3}$, $0$, and $\frac{2\pi}{3}$ such that $d_1 = d_2 = d_3$.

Then $r_N = 0$ for $N$ not a multiple of $3$, so the numerator of (1) is 0 for all $N$. 
Example 2

**Figure:** $\delta^2$ applied to partial sums of a function with equal jumps $(d_1 = d_2 = d_3 = 1)$.

**Figure:** $\delta^2$ applied to partial sums of a function with unequal jumps $(d_1 = -3, d_2 = 2, d_3 = 1)$. 
Investigate:

- Functions with a finite number of jumps anywhere in the interval \([-\pi, \pi]\).
- Functions with an infinite number of jumps.
- Other non-linear transforms applied to Fourier series.
Directions for Future Research

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References I


