A Torelli theorem for special divisor varieties X associated to doubly covered curves $\tilde{C}/C$

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0. Table of Contents

1. Introduction: In his famous paper [An] Andreotti introduced the following statement of the Torelli theorem for Jacobians in terms of the divisor varieties which parametrize the theta divisors: two smooth connected curves C_1, C_2 of genus g are birationally equivalent if and only if their symmetric products $C_1^{(g-1)}$, $C_2^{(g-1)}$ are birationally equivalent. Since isomorphism of the theta divisors $\Theta(C_1)$, $\Theta(C_2)$ implies birational equivalence of the symmetric products $C_1^{(g-1)}$, $C_2^{(g-1)}$ by Riemann's theorem, and since the smooth curves C_1, C_2 are birational if and only if they are biregularly isomorphic, Torelli's theorem that the C_i are isomorphic if and only if the $\Theta(C_i)$ are isomorphic is a corollary; indeed it follows that the C_i are isomorphic if and only if the $\Theta(C_i)$ are birational. In particular birational and biregular equivalence coincide for the symmetric products. The analogue of the symmetric product, for the Prym variety (P,Ξ) of a double cover $\tilde{C}/C$ of a smooth connected curve C of genus g, is the special divisor variety $X \subset \tilde{C}^{(2g-2)}$ of effective, even, precanonical divisors D on $\tilde{C}$, i.e. divisors D in $\tilde{C}^{(2g-2)}$ with Nm(D) in $|\omega_C|$ and $h^0(D)$ even. The varieties X are irreducible and normal when C is non hyperelliptic by [Be]. Since X is generically a $\mathbb{P}^1$ bundle over Ξ, biregular isomorphism of two Prym theta divisors Ξ_1, Ξ_2 implies birational equivalence of the varieties X_1, X_2. Hence if birational equivalence of the X_i were to imply isomorphism of the double covers $\tilde{C}_i/C_i$, then Torelli's theorem for Prym varieties would follow. Since counterexamples to the Prym Torelli problem [Do, Ve1] are known for doubly covered curves C of Clifford index ≤ 2, this result cannot hold for the special divisor varieties of such double covers. In view of Donagi's conjecture [Do, LS, cf. SV5], it is plausible however that it should hold for doubly covered curves of Clifford index ≥ 3. In this paper we consider the question of what is determined by the biregular isomorphism class of the divisor variety X. We show that for any two doubly covered non hyperelliptic curves $\tilde{C}_i/C_i$, if $\dim.\mathrm{sing}\Xi_i < \dim(P_i)-4$, (i.e. if $\tilde{C}_i/C_i$ are "not on Mumford's list"), then biregular isomorphism of the divisor varieties X_i does imply isomorphism of the double covers $\tilde{C}_i/C_i$. The method of recovering $\tilde{C}/C$ from X generalizes the construction used in [NR] to classify (even stable) rank 2 vector bundles on curves of genus two (and generalized later to higher rank and genus). Briefly we show X determines Ξ, then represent X as

$\mathbb{P}(\mathcal{E})$ for a suitable "rank 2 sheaf" $\mathcal{E}$ on Ξ, define $NR(\mathcal{E}) = \{\tau \text{ in } Pic^0(\mathcal{E}) : h^0(\mathcal{E} \otimes \tau) \neq 0\}$, and check this invariant is independent of choice of $\mathcal{E}$, up to translation in $Pic^0(\Xi)$. Then we show $Pic^0(\Xi) \cong P$, the Prym variety of $\tilde{C}/C$, and compute for a convenient choice $\mathcal{E}_p$ associated to a point p of $\tilde{C}$, that $NR(\mathcal{E}_p)$ corresponds under this isomorphism to $a_p(\tilde{C})$ = the Abel Prym model of $\tilde{C}$ in P. Finally, $a_p(\tilde{C})$ determines the double cover $\tilde{C}/C$.

In particular our result holds for all doubly covered C with $Cliff(C) \geq 3$. This means Prym - Torelli for these curves can fail only if two such divisor varieties X_i can be birationally isomorphic, have the same Albanese image, and yet be biregularly distinct. A corollary of our result is that Donagi's conjecture is equivalent to the assertion that two special divisor varieties $X(\tilde{C}_1/C_1)$, $X(\tilde{C}_2/C_2)$, parametrizing the same theta divisor Ξ, where $Cliff(C_i) \geq 3$, are not only birationally but also biregularly isomorphic.

Section 2 of this paper contains definitions and conventions about Prym varieties for the reader's convenience. The statement and outline of proof of the main result are in section 3, and the details of the proof are in sections 4 through 8. To the best of our knowledge, the first argument of this type, recovering a curve from a Narasimhan Ramanan invariant for bundles over the Jacobian of that curve, was by Kempf, in [Ke1; Ke2, cor 4.4.c, p.253].

2. Definition of the Prym variety (P,Ξ) and the divisor variety X

2.1 The Prym variety

Given a connected étale double cover $\pi:\tilde{C}\to C$ of a smooth non hyperelliptic curve C of genus $g \geq 3$, the kernel of the associated norm map $Nm:Pic^0(\tilde{C})\to Pic^0(C)$ on line bundles has two connected components $Nm^{-1}(0) = P_0 \cup P_1$, [Mu, bottom p.329, where Nm = ψ, cf. (b), bottom p.341]. If P_0 is the component containing 0, then the principal polarization of the Jacobian of $\tilde{C}$, considered as a cohomology class $\tilde{\vartheta}$ in $H^2(Pic^0(\tilde{C}),\mathbb{Z})$, restricts on P_0 to twice a principal polarization ξ, i.e. $\tilde{\vartheta}|P_0 = 2\xi$. The resulting pair (P_0,ξ) is by definition the principally polarized Prym variety determined by π. Since C has genus g, the double cover $\tilde{C}$ has genus 2g-1, and since Nm is surjective P_0 has dimension $p = g-1$. If $\iota:\tilde{C}\to\tilde{C}$ is the fix point free involution associated to the double cover π, then for each point p on $\tilde{C}$ let $p' = \iota(p)$, and define the Abel Prym map $a_p:\tilde{C}\to P_0$ associated to a point p on $\tilde{C}$ by $a_p(q) = a(q,p) = (1-\iota)(q-p) = q-p-q'+p'$ This is an embedding for non hyperelliptic C.

2.2 The Prym theta divisor

The polarized Prym variety (P_0,ξ) has a distinguished "theta divisor" determined up to translation, which may be described as follows. The inverse image $Nm^{-1}(\omega_C)$ of the canonical line bundle ω_C of C, under the norm map $Nm:Pic^{2g-2}(\tilde{C})\to Pic^{2g-2}(C)$ has again two connected components [Mu, pp. 341-2], $Nm^{-1}(\omega_C) = P \cup P^-$, where P = {L: $Nm(L) = \omega_C$ and $h^0(L)$ is even} and P^- = {L: $Nm(L) = \omega_C$ and $h^0(L)$ is odd}. Since $2g-2 = \tilde{g}-1$, the image of the Abel map $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to Pic^{2g-2}(\tilde{C})$ is the natural model $\tilde{\Theta}\subset Pic^{2g-2}(\tilde{C})$ of the theta divisor for the Jacobian of $\tilde{C}$, and by [Mu, Prop.(a), p.342], $P.\tilde{\Theta}$

$= 2\Xi$, where Ξ is a natural model for the theta divisor of the Prym variety determined by $\tilde{C}/C$. Analogously to representing the Jacobian variety $(\mathrm{Pic}^0(\tilde{C}),\tilde{\vartheta})$ by the pair $(\mathrm{Pic}^{2g-2}(\tilde{C}), \tilde{\Theta})$, we may consider the pair (P,Ξ) to represent the Prym variety (P_0,ξ).

2.3 The divisor variety X defined by $\tilde{C}/C$

We define the divisor variety as $X = \tilde{\alpha}^{-1}(P)$, where $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to \mathrm{Pic}^{2g-2}(\tilde{C})$ is the Abel map parametrizing the theta divisor of $\tilde{C}$, and $P\subset \mathrm{Pic}^{2g-2}(\tilde{C})$ is the "even" half of the set of precanonical line bundles on $\tilde{C}$ with respect to π, i.e. [We1, p.99] $X = \{D$ in $\tilde{C}^{(2g-2)}$ such that $\mathrm{Nm}(\mathcal{O}(D)) \cong \omega_C$ and $h^0(\tilde{C},\mathcal{O}(D))$ is even$\}$, is the set of effective, even, "precanonical" divisors on $\tilde{C}$ with respect to π. When C is non hyperelliptic X is normal and irreducible [Be, Cor. of prop.3, p.365]. Denoting by $\varphi:X\to P$ the restriction of the Abel map $\tilde{\alpha}$ to X, the image set $\Xi = \varphi(X) \subset P$ of effective even precanonical line bundles on $\tilde{C}$ with respect to π, is the natural model of the theta divisor for the Prym variety P.
By Riemann's singularities theorem $\Xi\subset \mathrm{sing}\tilde{\Theta}$, and since $P\cdot\tilde{\Theta} = 2\Xi$, all smooth points of Ξ are double points of $\tilde{\Theta}$. Hence by Abel's theorem the fiber of the Abel map $\tilde{\alpha}:\tilde{C}^{(2g-2)}\to \mathrm{Pic}^{2g-2}(\tilde{C})$ over a smooth point of Ξ is isomorphic to $\mathbb{P}^1$, and by Grauert's theorem, X is a Zariski $\mathbb{P}^1$ bundle over Ξ_{sm}. Thus when C is non hyperelliptic, X is a normal irreducible variety fibered by φ over smooth points of Ξ by copies of $\mathbb{P}^1$, and over singular points of Ξ by projective spaces of odd dimension ≥ 1. (It can happen that $\varphi^{-1}(L) \cong \mathbb{P}^1$ for some singular points L of Ξ, by [Mu, Prop. bottom p. 343].

3. Statement of main Theorem, outline of proof

We will prove the following Torelli theorem for the divisor varieties X defined in section 2.

Theorem 3.1. If C is a smooth non hyperelliptic curve of genus $g \geq 3$, and $\pi:\tilde{C}\to C$ a connected étale double cover such that either Ξ is smooth or $\dim.\mathrm{sing}\Xi \leq \dim(P)-5$, then X determines the double cover $\pi:\tilde{C}\to C$.

Remark 3.2.1. Recall Mumford's list [Mu, p.344] of doubly covered curves of genus $g \geq 3$ which do not satisfy the hypotheses of Theorem 3.1. They are: hyperelliptic curves C of genus $g \geq 3$; curves C with $g \geq 5$ and either trigonal or bielliptic (i.e. with either a degree three map to $\mathbb{P}^1$ or a degree two map to an elliptic curve); curves C with g = 4 or 5 and possessing a line bundle L with $2L = K_C$, $h^0(C,L) \neq 0$ and even, and $h^0(C,L\otimes\eta)$ even, where η is the square - trivial line bundle on C corresponding to the double cover π; curves C with g = 6 and having a line bundle L with $2L = K_C$, and $h^0(C,L) \geq 3$ and odd and $h^0(C,L\otimes\eta)$ even. Note by [SV2, bottom p.366, if $\tilde{C}\to C$ is a connected etale double cover with both $\tilde{C}$,C hyperelliptic, and g(C) = 3, then Ξ is a smooth curve of genus 2, but $X \cong \Xi\times\mathbb{P}^1$ does not determine $\tilde{C}/C$.

Remark 3.2.2. One use we will make of the hypothesis that π is not on Mumford's list is that Ξ is then locally factorial [BD], which relieves us of the task of arguing that the various Weil divisors we construct are Cartier divisors. It also allows us to use results from [BD] in which this hypothesis is made, and to make the argument in Lemma 5.3. where it gives us the requisite amount of depth. It seems probable that the theorem is also true assuming only the hypothesis that C is not hyperelliptic.

3.3 Summary of the proof

3.3.1 The Narasimhan Ramanan invariant

Narasimhan Ramanan in [NR] defined an injective map which associates to a stable even rank two vector bundle $\mathcal{E}$ over a curve Ξ of genus two, an invariant $\tilde{C}$ in the family of curves (with involution) in the linear equivalence class $|2\Xi|$, lying in the Jacobian of Ξ. Using the fact that in this case $J(\Xi) \cong P(\tilde{C}/C)$, Verra's analysis [Ve2] of the fibers of the Prym map in this genus, and work of H. Yin [Yi], we showed in [SV2] that $\mathbb{P}(\mathcal{E}) \cong X(\tilde{C}/C)$, and that the NR invariant gives an inverse, for doubly covered non hyperelliptic curves C of genus three, to the assignment $\tilde{C}/C \mapsto X$ taking a double cover to its special divisor variety X, which in that case is a $\mathbb{P}^1$ bundle over the theta divisor $\Xi \subset J(\Xi) \cong P(\tilde{C}/C)$. It is well known that the generalization of this invariant to stable even vector bundles of rank two over non hyperelliptic curves C of higher genus, takes values in the linear system $|2\Theta(C)|$ in the Jacobian of C, and again classifies such vector bundles [BV]. If we consider the generic Prym $\mathbb{P}^1$ bundles $\varphi: X \to \Xi$ defined by the special divisor varieties X associated to those doubly covered curves $\tilde{C}/C$ "not on Mumford's list", we will define an analog of the NR invariant for these, this time with values in the set of curves of twice the "minimal" homology class in P, and show that for these double covers, this invariant again provides a left inverse to the assignment $\tilde{C}/C \mapsto X$

taking a double cover to its special divisor variety X, thus establishing injectivity of the latter assignment.

3.3.2 Outline of proof

First we will recover from X both the Prym variety (P,Ξ) and the Abel map $\varphi: X \to \Xi \subset P$, by applying Serre's criterion for an Albanese map to show P is the Albanese variety of X, φ is the Albanese map, and Ξ is the Albanese image of X. Next we show for any coherent reflexive "rank 2 sheaf" $\mathcal{E}$ on Ξ such that $X|\Xi_{sm} = \mathbb{P}(\mathcal{E}|\Xi_{sm})$ and $c_1(\mathcal{E}|\Xi_{sm}) = \xi|\Xi_{sm}$ (where $\xi = c_1(\mathcal{O}_\Xi(\Xi))$), that the "Narasimhan Ramanan" invariant $NR(\mathcal{E}) = \{\tau$ in $Pic^0(\Xi)$ such that $h^0(\tau \otimes \mathcal{E}) \neq 0\}$ is determined by X up to translation in $Pic^0(\Xi)$. Then we prove that associated to any point p of $\tilde{C}$ there is such a sheaf $\mathcal{E}_p$ for which $NR(\mathcal{E}_p)$ is naturally isomorphic via the isomorphism $P_0 \cong Pic^0(\Xi)$, to the "Abel Prym" model $a_p(\tilde{C})$ of $\tilde{C}$ in P_0. Hence X determines $NR(\mathcal{E}_p) \cong a_p(\tilde{C}) \cong \tilde{C}$ up to translation in $Pic^0(\Xi)$, in particular up to isomorphism. By an argument of Welters [We2] using Matsusaka's "strong" Torelli theorem [Ma] for curves, $a_p(\tilde{C})$ also determines the involution on $\tilde{C}$ hence the double cover $\pi: \tilde{C} \to C$. If the Prym canonical map $C \to \mathbb{P}^{g-2}$ has degree one, eg. if C has no $g^1{}_4$, then π can be recovered as the (normalization of the) Gauss map on $a_p(\tilde{C})$. The remainder of the paper is devoted to the details of the proof. Since we have already proved Theorem 3.1 in [SV2] when g = 3, we may assume $g \geq 4$ throughout the paper, although some arguments will be stated as well for g = 3. In section 4 we show that $\varphi: X \to \Xi \subset P$ is the Albanese map of Ξ. In section 5 we show the invariant NR(X) is uniquely defined up to isomorphism by $NR(\mathcal{E})$ for any suitable rank 2 sheaf $\mathcal{E}$ on Ξ as above. In section 6 we construct for each point p on $\tilde{C}$ a rank 2 sheaf $\mathcal{E}_p$ on Ξ with $\mathbb{P}(\mathcal{E}_p|\Xi_{sm}) \cong X|\Xi_{sm}$. In section 7 we compute that $c_1(\mathcal{E}_p|\Xi_{sm}) = \xi|\Xi_{sm}$. In section 8 we compute that $NR(\mathcal{E}_p) = a_p(\tilde{C})$, and complete the argument.

4. Recovering (P,Ξ) from X as an Albanese variety

Proposition 4.1. Assume that C is non hyperelliptic of genus $g \geq 4$. Then the morphism $\varphi: X \to \Xi \subset P$ is an Albanese map for X.

Proof: This follows from two lemmas:

Lemma 4.2. The maps $\varphi: X \to \Xi$ and $\Xi \to P$, and thus also their composition $\varphi: X \to P$, induce isomorphisms on $H_{1,\mathbb{Z}}$.

Proof: Since $\Xi \subset P$ is an ample divisor on a smooth projective variety of dimension ≥ 3, by the Lefschetz hyperplane theorem the inclusion $\Xi \to P$ induces an isomorphism $H_1(\Xi,\mathbb{Z}) \to H_1(P,\mathbb{Z})$. Since all the fibers of $\varphi: X \to \Xi$ are (complex) projective spaces, we claim that the induced map $\varphi_*: H_1(X,\mathbb{Z}) \to H_1(\Xi,\mathbb{Z})$ is an isomorphism. By universal coefficients it will suffice to show that the induced map $\varphi^*: H^1(\Xi,\mathbb{Z}/n) \to H^1(X,\mathbb{Z}/n)$ on 1st cohomology is an isomorphism for all n (including n = 0). The Leray spectral sequence for the map φ: $X \to \Xi$ and coefficient sheaf $\underline{\mathbb{Z}/n}$ on X provides an exact sequence (∗) $0 \to H^1(\Xi,\varphi_*(\underline{\mathbb{Z}/n})) \to H^1(X,\mathbb{Z}/n) \to H^0(\Xi,R^1\varphi_*(\underline{\mathbb{Z}/n}))$. However, the stalk of the sheaf $R^i\varphi_*(\underline{\mathbb{Z}/n})$ at a point $y \in \Xi$ is computable as $H^i(\varphi^{-1}(y),\mathbb{Z}/n)$ and here $\varphi^{-1}(y) \cong \mathbb{P}^r$ so $H^0(\varphi^{-1}(y),\mathbb{Z}/n) = \mathbb{Z}/n$ and $H^1(\varphi^{-1}(y),\mathbb{Z}/n) = 0$. Thus $\varphi_*(\underline{\mathbb{Z}/n}) \cong \underline{\mathbb{Z}/n}$ and $R^1\varphi_*(\underline{\mathbb{Z}/n}) = 0$, and the exact sequence (*) implies $H^1(\Xi,\mathbb{Z}/n) \to H^1(X,\mathbb{Z}/n)$ is an isomorphism. **QED.**

Lemma 4.3. If X is irreducible, reduced, P is an abelian variety, and $\varphi:X\to P$ induces an isomorphism $\varphi_*: H_1(X,\mathbb{Z}) \to H_1(P,\mathbb{Z})$, then φ is an Albanese map for X.

Proof: Serre [Se; Th. 5, p. 10; Th. 10, p. 19] has shown for any irreducible variety X, existence of an Albanese map $f: X \to A$ such that the natural map $f_*:H_1(X, \mathbb{Z}) \to H_1(A, \mathbb{Z})$ is surjective, and satisfying a universal mapping property for morphisms from X to abelian varieties. Thus, after changing f by a translation, there exists a homomorphism $h: A \to P$ such that $h\circ f$ is the morphism $\varphi:X \to P$. Now we have a composition $X \to A \to P$ inducing maps $H_1(X, \mathbb{Z}) \to H_1(A, \mathbb{Z}) \to H_1(P, \mathbb{Z})$ such that the first map is surjective and the composition is an isomorphism. Thus the first map is also injective, hence an isomorphism, as is the second map. then since the homomorphism $h: A \to P$ of abelian varieties induces an isomorphism $H_1(A, \mathbb{Z}) \to H_1(P, \mathbb{Z})$ on 1st homology, h must be an isomorphism. Since $h\circ f$ is the morphism $\varphi:X \to P$, thus $(h\circ f)(X) = \varphi(X) = \Xi$, so h induces an isomorphism from f(X) onto Ξ.

QED for Lemma 4.3 and also for Proposition 4.1.

5. Definition of the Narasimhan Ramanan invariant NR(X)

For any coherent sheaf $\mathcal{E}$ on Ξ, we could consider the set $\{\tau$ in $\mathrm{Pic}(\Xi)$ such that $h^0(\mathcal{E}\otimes\tau) \neq 0\}$ but it is not obvious in which component of $\mathrm{Pic}(\Xi)$ we find interesting information about $\mathcal{E}$, nor for which sheaves $\mathcal{E}$ the invariant gives useful information about Ξ. The most natural version of the original Narasimhan Ramanan invariant [NR] for vector bundles over a curve C is perhaps to consider (stable) bundles $\mathcal{E}$ with $\det(\mathcal{E}) = \omega_C$ and take $NR(\mathcal{E}) = \{\tau \text{ in } Pic^0(C) \text{ such that } h^0(\mathcal{E}\otimes\tau) \neq 0\}$. In particular it is crucial to choose correctly the (relationship between the) degree of $\mathcal{E}$ and the degree of the elements τ of $NR(\mathcal{E})$. A technical point in giving an generalization to the present setting is that X is a $\mathbb{P}^1$ bundle in general only over the smooth points of Ξ, so we need to check that with an appropriate global hypothesis, we can restrict attention to the part of X lying over the smooth points of Ξ. We also need to choose the right chern classes for our $\mathcal{E}$ and our τ. The appropriate generalization (which is analogous to the natural choices above) is as follows:

Given the Abel map $\varphi: X \to \Xi$, let $X|\Xi_{sm}$ denote $\varphi^{-1}(\Xi_{sm})$.

Definition 5.1. If $\tilde{C}/C$ is a double cover of a non hyperelliptic curve C, and $\mathcal{E}$ is any reflexive coherent sheaf on Ξ whose restriction to Ξ_{sm} is a rank 2 vector bundle satisfying the two conditions:

(a) $c_1(\mathcal{E}|\Xi_{sm}) = \xi|\Xi_{sm} \in H^2(\Xi_{sm},\mathbb{Z})$, and

(b) $\mathbb{P}(\mathcal{E}|\Xi_{sm}) \cong X|\Xi_{sm}$.

Then define **NR($\mathcal{E}$)** $= \{\tau \text{ in } Pic^0(\Xi) \text{ such that } h^0(\mathcal{E}\otimes\tau) \neq 0\}$.

Next we ask how much this invariant depends on the choice of $\mathcal{E}$.

Proposition 5.2. Assume $\tilde{C}/C$ satisfies the hypotheses of Thm.3.1 and $\mathcal{E}_1,\mathcal{E}_2$ are sheaves on Ξ with the properties of Definition 5.1. Then $\mathcal{E}_1 \cong \overline{\mathcal{M}}\otimes\mathcal{E}_2$ for some line bundle $\overline{\mathcal{M}}$ in $Pic^0(\Xi)$.

Proof: This will follow from the next lemma.

Lemma 5.3. If $\tilde{C}/C$ and $\mathcal{E}_1, \mathcal{E}_2$ satisfy the hypotheses of Prop. 5.2, then: **(i)** $\mathcal{E}_1|\Xi_{sm} \cong (\mathcal{E}_2|\Xi_{sm}) \otimes \mathcal{M}$ for some line bundle $\mathcal{M}$ on Ξ_{sm} and, for any such $\mathcal{M}$, $2c_1(\mathcal{M}) = 0$ in $H^2(\Xi_{sm}, \mathbb{Z})$.

(ii) If any $\mathcal{M}$ as in (i) extends to a line bundle $\overline{\mathcal{M}}$ on Ξ, then $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \overline{\mathcal{M}}$ on Ξ and $\overline{\mathcal{M}} \in \mathrm{Pic}^0(\Xi)$ (equivalently, $c_1(\overline{\mathcal{M}}) = 0$ in $H^2(\Xi, \mathbb{Z})$).

(iii) In fact there exists $\overline{\mathcal{M}} \in \mathrm{Pic}^0(\Xi)$ such that $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \overline{\mathcal{M}}$ on Ξ.

Proof: (i) Since $\mathcal{E}_1|\Xi_{sm}$ and $\mathcal{E}_2|\Xi_{sm}$ are vector bundles on Ξ_{sm} yielding the same projective bundle (i.e. $\mathbb{P}(\mathcal{E}_1|\Xi_{sm}) \cong X|\Xi_{sm} \cong \mathbb{P}(\mathcal{E}_2|\Xi_{sm})$, and the isomorphisms commute with the projections to Ξ_{sm}), it is a standard exercise that $\mathcal{E}_1|\Xi_{sm} \cong (\mathcal{E}_2|\Xi_{sm}) \otimes \mathcal{M}$ for some line bundle $\mathcal{M}$ on Ξ_{sm}, [cf. Ha, p.171, ex. 7.10d, but note the opposite convention there, p.162, on $\mathbb{P}(\mathcal{E})$]. Now take the 1^{st} chern class of both sides: in $H^2(\Xi_{sm}, \mathbb{Z})$, $c_1(\mathcal{E}_1|\Xi_{sm}) = c_1((\mathcal{E}_2|\Xi_{sm}) \otimes \mathcal{M}) = c_1(\mathcal{E}_2|\Xi_{sm}) + 2c_1(\mathcal{M})$ since $\mathcal{E}_2|\Xi_{sm}$ is a rank 2 vector bundle. Then, since $c_1(\mathcal{E}_1|\Xi_{sm}) = \xi|\Xi_{sm} = c_1(\mathcal{E}_2|\Xi_{sm})$, we get $2c_1(\mathcal{M}) = 0$.

(ii) Assuming that $\mathcal{M}$ extends to a line bundle $\overline{\mathcal{M}}$ on Ξ, then $\mathcal{E}_1$ and $\mathcal{E}_2 \otimes \overline{\mathcal{M}}$ are both reflexive coherent sheaves on Ξ, and there is an isomorphism between the restrictions $\mathcal{E}_1|\Xi_{sm}$ and $(\mathcal{E}_2 \otimes \overline{\mathcal{M}})|\Xi_{sm}$ ($\cong (\mathcal{E}_2|\Xi_{sm}) \otimes \mathcal{M}$). Therefore, by the unique extension property of sections of these sheaves (using depth ≥ 2 along $\mathrm{Sing}(\Xi)$ because C is non h.e. and $g(C) \geq 3$, [cf. SV4, proof of Lemma (4.5)]), there is an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2 \otimes \overline{\mathcal{M}}$ on Ξ.

Next consider $\alpha = c_1(\overline{\mathcal{M}}) \in H^2(\Xi, \mathbb{Z})$. We want to show that $\alpha = 0$ (since $\mathrm{Pic}^0(\Xi) = \ker\{c_1 : \mathrm{Pic}(\Xi) \to H^2(\Xi, \mathbb{Z})\}$, and we then conclude that $\overline{\mathcal{M}} \in \mathrm{Pic}^0(\Xi)$). In fact it will suffice to show that $2\alpha = 0$ since

$H^2(\Xi,\mathbb{Z})$ is torsion free. [Indeed, when $\dim(\Xi) > 2$, i.e. when C has genus $g \geq 5$, the Lefschetz theorem gives immediately an isomorphism $H^2(\Xi,\mathbb{Z}) \cong H^2(P,\mathbb{Z})$, which is torsion free. When Ξ has dimension 2 or 1 then (P,Ξ) is the Jacobian $J(\Sigma)$ of a curve Σ of genus $p = 3$ or 2, and we can give a direct argument: if $p = 2$, then Ξ is a (smooth) genus 2 curve and $H^2(\Xi,\mathbb{Z}) \cong \mathbb{Z}$. If $p = 3$, then Ξ is either the symmetric product $\Sigma^{(2)}$ (if Σ is nonhyperelliptic) or $\Sigma^{(2)}$ with the $g^1{}_2$ collapsed to a point if Σ is hyperelliptic. In either case, $H^2(\Xi,\mathbb{Z})$ injects (e.g. use the Leray s.s. in the hyperelliptic case) into $H^2(\Sigma^{(2)},\mathbb{Z})$ which is torsion free (by Macdonald's results on the cohomology of symmetric products).] Thus it remains to show that if $\alpha = c_1(\overline{\mathcal{M}}) \in H^2(\Xi,\mathbb{Z})$ then $2\alpha = 0$. We know that 2α restricts to zero in $H^2(\Xi_{sm},\mathbb{Z})$ (since $\overline{\mathcal{M}}^{\otimes 2}$ restricts to $\mathcal{M}^{\otimes 2}$ and $c_1(\mathcal{M}^{\otimes 2}) = 2c_1(\mathcal{M}) = 0$). Thus, consider $\overline{\mathcal{M}}^{\otimes 2} \in \mathrm{Pic}(\Xi)$ and the diagram:

$$\begin{array}{ccc} H^1(\Xi,\mathcal{O}) & \to & \mathrm{Pic}(\Xi) \\ \downarrow & & \downarrow \\ H^1(\Xi_{sm},\mathcal{O}) & \to & \mathrm{Pic}(\Xi_{sm}) \end{array}$$

Since $2c_1(\mathcal{M}) = 0$ in $H^2(\Xi_{sm},\mathbb{Z})$, by exactness of the exponential sequence $\mathcal{M}^{\otimes 2} \in \mathrm{im}\{H^1(\Xi_{sm},\mathcal{O}) \to \mathrm{Pic}(\Xi_{sm})\}$. Moreover the left vertical map $(H^1(\Xi,\mathcal{O}) \to H^1(\Xi_{sm},\mathcal{O}))$ is an isomorphism. [This follows from the exact sequence of local cohomology $H^1{}_{\mathrm{sing}\Xi}(\mathcal{O}) \to H^1(\Xi,\mathcal{O}) \to H^1(\Xi_{sm},\mathcal{O}) \to H^2{}_{\mathrm{sg}\Xi}(\mathcal{O})$ and the vanishing of the two groups $H^1{}_{\mathrm{sing}\Xi}(\mathcal{O}) = H^2{}_{\mathrm{sing}\Xi}(\mathcal{O}) = 0$. These in turn vanish by

[Gr, Prop. 1.4, p.5], since as long as singΞ has codimension ≥ 3 in Ξ, the depth of $\mathcal{O}_\Xi$ along singΞ is ≥ 3 which implies vanishing of the local cohomology sheaves, which by the spectral sequence in [Gr, Prop. 1.4]. implies vanishing of the local cohomology groups above.] Therefore we can modify $\overline{\mathcal{M}}^{\otimes 2}$ by an element β of $\mathrm{Pic}^0(\Xi)$ to get an element $\mu = (\overline{\mathcal{M}}^{\otimes 2}) \otimes \beta^{-1} \in \mathrm{Pic}(\Xi)$ such that $\mu|\Xi_{sm}$ is trivial in $\mathrm{Pic}(\Xi_{sm})$. But then μ must be trivial in $\mathrm{Pic}(\Xi)$ since if a Cartier divisor D on Ξ becomes (complex analytically) principal on Ξ_{sm}, i.e. the divisor of a meromorphic function f on Ξ_{sm}, then since Ξ is normal, f is the restriction of a (unique) meromorphic function $\overline{f}$ on Ξ and D is the divisor of $\overline{f}$. Therefore, $\overline{\mathcal{M}}^{\otimes 2} = \mu \otimes \beta = \beta \in \mathrm{Pic}^0(\Xi)$, so $2\alpha = c_1(\overline{\mathcal{M}}^{\otimes 2}) = 0$ in $H^2(\Xi, \mathbb{Z})$.

(iii) Let $\mathcal{M}$ be any (algebraic) line bundle on Ξ_{sm} such that $\mathcal{E}|\Xi_{sm} \cong (\mathcal{E}_p|\Xi_{sm}) \otimes \mathcal{M}$. Since for π not on Mumford's list we have $\mathrm{codim}_\Xi(\mathrm{Sing}(\Xi)) \geq 4$, hence Ξ is locally factorial. Then there exists a line bundle $\overline{\mathcal{M}}$ on Ξ extending $\mathcal{M}$. Indeed, represent $\mathcal{M}$ by a (Cartier) divisor D on Ξ_{sm} and then take the closure $\overline{D}$ of D in X as a Weil divisor. Since Ξ is locally factorial, $\overline{D}$ is Cartier on Ξ and hence $\overline{\mathcal{M}} = \mathcal{O}_\Xi(\overline{D})$ is a line bundle on Ξ extending $\mathcal{O}_{\Xi_{sm}}(D) = \mathcal{M}$. Then from part (2) it follows that $\mathcal{E} \cong \mathcal{E}_p \otimes \overline{\mathcal{M}}$ on Ξ and $\overline{\mathcal{M}} \in \mathrm{Pic}^0(\Xi)$. **Q.E.D. for Lemma 5.3 and hence for Proposition 5.2.**

Lemma 5.4. If $\tilde{C}/C$ and $\mathcal{E}$ satisfy the hypotheses of Prop. 5.2, then X determines $NR(\mathcal{E})$ up to translation in $\mathrm{Pic}^0(\Xi)$.

Proof: If $\mathcal{E}_1, \mathcal{E}_2$ are two reflexive coherent sheaf on Ξ whose restrictions to Ξ_{sm} are rank 2 vector bundles satisfying the two

conditions $c_1(\mathcal{E}_i) = \xi$, and $\mathbb{P}(\mathcal{E}_i) \cong X$, then by Proposition 5.2 above, $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \overline{\mathcal{M}}$, for some $\overline{\mathcal{M}}$ in $\mathrm{Pic}^0(\Xi)$, hence $NR(\mathcal{E}_1) \otimes \overline{\mathcal{M}} = NR(\mathcal{E}_2)$.
QED.

Definition 5.5. If $\tilde{C}/C$ and $\mathcal{E}$ satisfy the hypotheses of Prop. 5.2 and X is the associated divisor variety, then define $NR(X) = NR(\mathcal{E})$, taken only up to translation in $\mathrm{Pic}^0(\Xi)$.

Remark 5.6. So far we have shown the uniqueness but not the existence of the invariant NR(X), because we have not yet proved sheaves $\mathcal{E}$ exist satisfying the hypotheses of Prop. 5.2. In the next section, for each choice of a point p in $\tilde{C}$ we will produce a rank 2 sheaf $\mathcal{E}_p$ with the desired properties, by pushing down a restricted Poincare line bundle for $\tilde{C}$ over Ξ. It will follow from the computation of $NR(\mathcal{E}_p)$ in the subsequent sections that just as in the case of rank 2 stable bundles over curves, the invariant NR(X) classifies those varieties X arising from $\tilde{C}/C$ not on Mumford's list, in the sense that X (modulo isomorphism) and NR(X) (modulo translation) are equivalent data.

6. Construction of effective Poincare line bundles $\mathcal{L}_p$ on $\tilde{C}\times\Xi$

6.1 Motivation

We want to prove the existence of a sheaf $\mathcal{E}$ satisfying the properties in Proposition 5.2, i.e. a coherent reflexive sheaf $\mathcal{E}$ on Ξ whose restriction to Ξ_{sm} is a rank 2 vector bundle such that $c_1(\mathcal{E}|\Xi_{sm}) = \xi|\Xi_{sm} \in H^2(\Xi_{sm},\mathbb{Z})$, and $\mathbb{P}(\mathcal{E}|\Xi_{sm}) \cong X|\Xi_{sm}$. Since for L a smooth point of Ξ, the fiber $\varphi^{-1}(L)$ in X is isomorphic to the pencil $|L|$, we would have $\mathbb{P}(\mathcal{E}|\Xi_{sm}) \cong X|\Xi_{sm}$ (by Grauert's theorem) if $\mathcal{E}$ were a vector bundle over Ξ_{sm} with fiber $H^0(\tilde{C},L)$ over L. This in turn would be true if $\mathcal{L}$ were a "Poincare line bundle" on $\tilde{C} \times \Xi$ such that $\mathcal{L}|\tilde{C}\times\{L\} \cong L$, and we set $\mathcal{E} \cong \pi_*(\mathcal{L})$ where $\pi:\tilde{C}\times\Xi\to\Xi$ is the projection. Thus we want to look for suitable Poincare bundles $\mathcal{L}$ on $\tilde{C}\times\Xi$. Since furthermore $\pi_*(\mathcal{L})$ is effective if and only if $\mathcal{L}$ is effective, and $NR(\pi_*(\mathcal{L}))$ is defined in terms of effective twists of $\pi_*(\mathcal{L})$, computing $NR(\pi_*(\mathcal{L}))$ will boil down to finding effective Poincare bundles (with given chern class), so it is natural to begin by looking for effective Poincare divisors on $\tilde{C}\times\Xi$. [Assuming $\tilde{C}$ satisfies the hypotheses of Thm.3.1 makes this easier since then Ξ and hence $\tilde{C}\times\Xi$ are locally factorial (see Remark 3.2.2), so we can specify Cartier divisors by giving the underlying Weil divisor. This does not seem to be crucial in what follows, but it is convenient.] We will define next an effective Poincare divisor S_p by pulling back a family $\mathcal{D}_p$ of "hyperplanes" in the pencils |L|, by the universal family γ of rational maps defined by the line bundles L in Ξ.

6.2 The fundamental divisors $\mathcal{D}_p \subset X$

For any point p in $\tilde{C}$, if $\tilde{C}^{(2g-2)} \supset \mathbb{D}_p = \{D \text{ in } \tilde{C}^{(2g-2)} : D \geq p\}$, then $\mathbb{D}_p$ is an ample Cartier divisor on $\tilde{C}^{(2g-2)}$. (The pullback of $\mathbb{D}_p$ to the cartesian product $\Pi\tilde{C}$, is the tensor product of the pullbacks of the ample bundles $\mathcal{O}_{\tilde{C}}(p)$ on each factor.)

Definition 6.2.1. For any point p in $\tilde{C}$, define an ample Cartier divisor $\mathcal{D}_p$ on X, by $X \supset \mathcal{D}_p = \{D \text{ in } X \subset \tilde{C}^{(2g-2)} : D \geq p\} = \mathbb{D}_p \cdot X$.

Lemma 6.2.2. If C is nonhyperelliptic of genus $g \geq 4$, $\tilde{C}/C$ is any double cover satisfying the hypotheses of Thm. 3.1, and p is any point of $\tilde{C}$, then the divisor $\mathcal{D}_p$ on X is reduced and irreducible.

Proof: Assume first $g(C) \geq 5$. Then since $\mathcal{D}_q$ is ample for all q, the irreducibility of $\mathcal{D}_p$ follows from the irreducibility and reducedness of the intersection $\mathcal{D}_p \cap \mathcal{D}_q$ for $p \neq q$, proved in [BD, p.615] whenever C is neither hyperellptic, trigonal, nor a double cover of an elliptic curve. If g = 4, the irreducibility and reducedness of $\mathcal{D}_p \cap \mathcal{D}_q$ for q general is proved in Claim 6.4.9 below, so $\mathcal{D}_p$ is irreducible if g = 4 as well.

Alternately the method of [Be, Prop.3, p.365], analyzing the ramification and degree of the map defined by the linear series $|K_C - \pi(p) - \pi(q)|$, can be applied directly to the irreducibility of $\mathcal{D}_p$ by analyzing the map defined by $|K_C - \pi(p)|$. In fact $\mathcal{D}_p$ can be shown to be irreducible for all connected etale double covers of non hyperelliptic C with $g(C) \geq 3$ and any point p on $\tilde{C}$. **QED.**

Remark 6.2.3. The divisors $\mathcal{D}_p$ provide distinguished quasi sections of the Abel map $\varphi: X \to \Xi$ in the sense that the restriction $\varphi_p: \mathcal{D}_p \to \Xi$ is a birational morphism. I.e. since the fibers of φ are projective spaces of dimension ≥ 1 and it is one condition to contain p, φ_p is a surjective morphism of varieties of the same dimension whose fibers are projective spaces, hence in general single points. Since $\mathcal{D}_p$ is reduced and irreducible, $\mathcal{D}_p$ meets a general fiber of φ in one reduced point, and thus we may consider $\mathcal{D}_p$ generically as a family

of "hyperplanes" in the family of pencils |L| for L in Ξ.
We will construct a natural Poincare divisor by pulling back $\mathcal{D}_p$ by the universal rational map γ on $\tilde{C}\times\Xi$ discussed in the next section.

6.3 The universal rational map $\gamma:\tilde{C}\times\Xi$--->X

Consider the map $\gamma:\tilde{C}\times\Xi$--->X which is the disjoint union over Ξ of the rational projective maps $\varphi_L:\tilde{C}$--->|L| defined by those points L of Ξ which define pencils. [Note that since |L| is a pencil, |L| and |L|* are isomorphic.] If $\tilde{C}\times\Xi \supset G$ = {(q,L): |L| is a pencil, and q is not a base point of |L|} then γ is a morphism on G defined by γ(q,L) = the unique divisor D in |L| containing q. We claim the complement of G in $\tilde{C}\times\Xi$ has codimension $\geq$ 2. If we fix q, the set Z_q of those L in Ξ for which q is a base point is the image of the exceptional locus of $\mathcal{D}_q$ under the birational map $\varphi_q:\mathcal{D}_q\rightarrow\Xi$, hence Z_q has codimension $\geq$ 2 in Ξ, since Ξ is normal when C is non hyperelliptic. Since the locus of pairs (q,L) for which q is a base point of L meets each fiber $q\times\Xi$ in a set of codimension $\geq$ 2 in Ξ, this locus of pairs has codimension $\geq$ 2 in $\tilde{C}\times\Xi$. Moreover the set of L's such that L is not a pencil lie in the singular locus of Ξ, and thus Ξ normal again implies the set of pairs {(q,L): L not a pencil} has codimension $\geq$ 2 in $\tilde{C}\times\Xi$. Thus the "bad" locus B = $\tilde{C}\times\Xi$ - G, where γ is not defined by the prescription above, namely B = {pairs of form (q,L) such that either q is a base point of |L|, or |L| is not a pencil}, has codimension $\geq$ 2 in $\tilde{C}\times\Xi$.
Since each Z_q has codimension $\geq$ 2 in Ξ, the set Z = $\cup_q Z_q$ = {line bundles L in Ξ with at least one base point} has codimension $\geq$ 1 in Ξ. Thus Ξ normal implies W = $Z\cup\mathrm{sing}\Xi$ has codimension $\geq$ 1 in Ξ. If

$U = \Xi - W$, then $X \supset \varphi^{-1}(U) \to U$ is a $\mathbb{P}^1$ bundle and $\tilde{C} \times U \subset G$, i.e. $\gamma : \tilde{C} \times U \to \varphi^{-1}(U) \subset X$ is a morphism. Since for L in U, $\gamma_L : \tilde{C} \to |L|$ has degree 2g-2, both the morphism $\gamma : \tilde{C} \times U \to \varphi^{-1}(U)$ and the rational map $\gamma : \tilde{C} \times \Xi \dashrightarrow X$ have degree 2g-2.

6.4 The effective Poincare divisor $S_p \subset \tilde{C} \times \Xi$

Choose any point p of $\tilde{C}$. Let $\gamma : \tilde{C} \times \Xi \dashrightarrow X$ be the rational map defined in the previous section with domain of definition G, and $\mathcal{D}_p \subset X$ the fundamental reduced irreducible divisor from section 6.2.

Definition 6.4.1. Define the pullback Cartier divisor $G \supset \gamma^*(\mathcal{D}_p)$ (as a set) = $\{(q,L)$ in G : $\gamma(q,L)$ is in $\mathcal{D}_p\}$, and let S_p be the closure in $\tilde{C} \times \Xi$ of the Cartier divisor $\gamma^*(\mathcal{D}_p)$; i.e. take the closure of each component, with the same multiplicity.

Remark 6.4.2. Since $\tilde{C} \times \Xi$ is non singular in codimension 3 for $\tilde{C}/C$ not on Mumford's list, the closure in $\tilde{C} \times \Xi$ of each component of the pullback divisor $\gamma^*(\mathcal{D}_p)$ is a Cartier divisor in $\tilde{C} \times \Xi$.

We want to describe the divisor S_p geometrically. We claim that as a set S_p = {all (q,L) in $\tilde{C} \times \Xi$ such that L belongs to $\varphi(\mathcal{D}_p \cap \mathcal{D}_q)$}. More precisely S_p is reduced and has exactly two irreducible components: $\{p\} \times \Xi$ and the closure in $\tilde{C} \times \Xi$ of the irreducible set of all {(q,L) such that: $p \neq q$ and L belongs to $\varphi(\mathcal{D}_p \cap \mathcal{D}_q)$}. This will be proved in the next two lemmas. (see also Remark 6.4.11 below.)

Lemma 6.4.3. $\gamma^*(\mathcal{D}_p)$ is reduced, hence also S_p is reduced.

Proof: If G is the domain of regularity of γ and $U = \Xi - W$ is the open subset of line bundles L in Ξ_{sm} which have no base points, recall that $\tilde{C}\times U\subset G$, and $\gamma:\tilde{C}\times U\to\varphi^{-1}(U)\subset X$ is a morphism such that $\varphi^{-1}(U)\to U$ is a $\mathbb{P}^1$ bundle, and for every L in U, $\gamma_L:\tilde{C}\to|L|$ is a morphism of degree 2g-2.

Claim 6.4.4. $\gamma^*(\mathcal{D}_p)\cap(\tilde{C}\times U)$ is dense in every component of $\gamma^*(\mathcal{D}_p)$.

Proof: For this it suffices to show that no component of $\gamma^*(\mathcal{D}_p)$ is contained in $\tilde{C}\times(\Xi-U)$. Since $\Xi-U$ is a disjoint union $\Xi-U = \text{sing}\Xi \cup (Z\cap\Xi_{sm})$, and $\text{sing}\Xi$ has codimension ≥ 2 in Ξ, it suffices to show no component of $\gamma^*(\mathcal{D}_p)$ is contained in $\tilde{C}\times(Z\cap\Xi_{sm})$. Since Z_p also has codimension ≥ 2 in Ξ, it suffices to show this for $\tilde{C}\times(\{Z-Z_p\}\cap\Xi_{sm})$. For this we compute the dimension of $\gamma^*(\mathcal{D}_p)\cap(\tilde{C}\times(\{Z-Z_p\}\cap\Xi_{sm}))$ by projecting onto the factor $(\{Z-Z_p\}\cap\Xi_{sm})$. Since for L in this set, |L| is a pencil for which p is not a base point, only one divisor of |L| contains p. Hence $\gamma(q,L)$ cannot lie in $\mathcal{D}_p$ unless q is in the finite support of that divisor. Hence $\gamma^*(\mathcal{D}_p)\cap(\tilde{C}\times(\{Z-Z_p\}\cap\Xi_{sm}))$ projects finitely onto the factor $(\{Z-Z_p\}\cap\Xi_{sm})$, hence the part of $\gamma^*(\mathcal{D}_p)$ in this intersection has dimension at most that of Z. Since $\dim(Z)\leq\dim(\tilde{C}\times\Xi) - 2$, no component of $\gamma^*(\mathcal{D}_p)$ lies outside of $\tilde{C}\times U$. **QED for Claim 6.4.4.**

Now we may restrict attention to the pullback of $\mathcal{D}_p$ by the finite morphism of degree 2g-2, $\gamma:\tilde{C}\times U\to\varphi^{-1}(U)$. It follows from this and Claim 6.4.4 that every component of $\gamma^*(\mathcal{D}_p)$ dominates $\mathcal{D}_p$. We will find an open dense set of $\varphi^{-1}(U)\cap\mathcal{D}_p$ over which γ is not ramified.

Claim 6.4.5. A general divisor D in $\mathcal{D}_p$, hence also in $\varphi^{-1}(U)\cap\mathcal{D}_p$, has no multiple points.

Proof: Consider the norm map $Nm:X\to|\omega_C|$ on divisors, and note that any point of X representing a divisor with a multiple point maps via Nm to a divisor in $|\omega_C|$ with a multiple point. Since Nm defines a finite map from $\mathcal{D}_p$ onto the hyperplane $Nm(\mathcal{D}_p)$ = $|\omega_C - \overline{p}| + \overline{p}$ = $H_{\overline{p}}$ in $|\omega_C|$, where $\overline{p} = \pi(p)$, it suffices to find an open dense subset of this hyperplane containing no multiple divisors. Since the canonical map on the non hyperelliptic curve C is an embedding, $|\omega_C - \overline{p}|$ is base point free, Hence the general divisor in $|\omega_C - \overline{p}|$ does not contain $\overline{p}$, and by Bertini the general divisor is not multiple. [Take a general hyperplane in canonical space through $\overline{p}$ to cut such a divisor.] Then every point in the preimage of that open dense subset of $|\omega_C - \overline{p}| + \overline{p}$ must consist of non multiple divisors. Since the preimage of that set is dense in $\mathcal{D}_p$ we have a dense open subset of $\mathcal{D}_p$ consisting of divisors of distinct points. **QED Claim 6.4.5.**

Claim 6.4.6. A general D in $\varphi^{-1}(U) \cap \mathcal{D}_p$, is not in the branch locus of γ.

Proof: We must show that a general D in $\varphi^{-1}(U) \cap \mathcal{D}_p$ is not the image under γ of a point of $\tilde{C} \times U$ at which the differential of γ has a non zero kernel. Now if L = $\mathcal{O}(D)$ and a tangent vector to $\tilde{C} \times U$ at (q,L) were killed by the differential of γ, composing with φ shows it must be killed by projection to U, hence must be tangent to $\tilde{C} \times \{L\}$. But then the restriction of γ to $\tilde{C} \times \{L\} \to |L|$ would be ramified at (q,L) hence branched over D. But this restriction is branched precisely over divisors D of |L| with multiple points. Since a general D as in Claim 6.4.5 has no multiple points, it is not in the branch locus of this restricted mapping. **QED Claim 6.4.6 and Lemma 6.4.3.**

Now that S_p is reduced, it suffices to describe it set theoretically.

Lemma 6.4.7. As a set, $\tilde{C}\times\Xi \supset S_p = \cup\{\{q\}\times\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$, for all q in $\tilde{C}\}$; i.e. S_p = {all (q,L) such that L belongs to $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)\}$. More precisely S_p has exactly two irreducible components: $\{p\}\times\Xi$ and the closure in $\tilde{C}\times\Xi$ of the irreducible set of all {(q,L) such that: $p \neq q$ and L belongs to $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)\}$.

Proof: Since we know by Claim 6.4.4 that S_p is the closure of $\gamma^*(\mathcal{D}_p)\cap(\tilde{C}\times U)$ in $\tilde{C}\times\Xi$, it suffices to describe the components of $\gamma^*(\mathcal{D}_p)\cap(\tilde{C}\times U)$ = {(q,L) where L in Ξ_{sm} is a pencil, q not a base point of L, and the unique divisor of |L| containing q also contains p}.

It is useful to consider, in addition to the set $\tilde{C}\times U$ on which γ is regular, also a complete blowup of $\tilde{C}\times\Xi$ on which γ becomes regular. Thus consider in $\tilde{C}\times X$ the incidence divisor I = {(q,D): $D \geq q$}, and the restriction to I of the "Abel map" $1\times\varphi:\tilde{C}\times X\to\tilde{C}\times\Xi$.

Claim 6.4.8. I is irreducible and maps birationally to $\tilde{C}\times\Xi$.

Proof: The proper projection $f:\tilde{C}\times X\to\tilde{C}$ onto $\tilde{C}$ fibers I by the fibers $f^{-1}(q) = \mathcal{D}_q$ = {D in X with $D\geq q$} which we have just seen in Lemma 6.2.2 are reduced and irreducible. Since the $\mathcal{D}_q$ are all divisors in X, hence all of the same dimension, I is irreducible, [Sh, Thm.8, p.61]. Now, over the set $\tilde{C}\times U$ the map $I\to\tilde{C}\times\Xi$ has singleton fibers hence is birational, since for each (q,L) in $\tilde{C}\times U$, there is a unique divisor D in X containing q. In fact since $\tilde{C}\times U$ is smooth, the map is an isomorphism over $\tilde{C}\times U$. **QED for Claim 6.4.8.**

Now if $\pi: I \to X$ is the restricted projection $\tilde{C} \times X \to X$ to the second factor, then π = the (unique regular extension of the) composition $\gamma \circ (1 \times \varphi): I \to \tilde{C} \times \Xi \dashrightarrow X$, and since π and γ differ by the isomorphism $1 \times \varphi$ over the open set $\varphi^{-1}(U) \subset X$, we have $\pi^*(\mathcal{D}_p) \supset (1 \times \varphi)^*(S_p)$. In fact we claim these are equal. To see this, we will show the right hand side has at least two irreducible components while the left hand side has at most two. Consider first the restriction I_p of the incidence divisor I to $\tilde{C} \times \mathcal{D}_p$; i.e. $I_p = \{(q,D): D \geq q \text{ and } D \geq p\}$ (NOTE: when $p \neq q$ this means $D \geq p+q$, but when $p = q$ it means only $D \geq p$). Then $I_p = \pi^{-1}(\mathcal{D}_p)$ under the projection $\pi: I \to X$. So the irreducible components of $\pi^*(\mathcal{D}_p)$ are precisely those of $I_p = \cup_q \{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q)$ $= \{p\} \times \mathcal{D}_p \cup \text{closure}(\cup_{q \neq p} \{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q))$. Denote by M_p the component $\text{closure}(\cup_{q \neq p} \{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q))$.

Claim 6.4.9. I_p has exactly two irreducible components: $\{p\} \times \mathcal{D}_p$ and $M_p = \text{closure}(\cup_{q \neq p} \{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q))$.

Proof: I_p is pure dimensional and if we project $I_p \subset \tilde{C} \times \mathcal{D}_p$ onto $\tilde{C}$, every irreducible component of I_p either dominates $\tilde{C}$ or maps to a single point. The inverse image of any point $q \neq p$ is $\{q\} \times (\mathcal{D}_p \cap \mathcal{D}_q)$, while the inverse image of p is $\{p\} \times \mathcal{D}_p$. The set $\{p\} \times \mathcal{D}_p$ is irreducible by Lemma 6.2.2 and thus is the only non dominating component of I_p. To show there is only one dominating component it suffices to find one point $q \neq p$ such that the fiber over q is reduced and irreducible. The sets $(\mathcal{D}_p \cap \mathcal{D}_q)$ for every $p \neq q$, are reduced and irreducible by [BD p.615, lines 13-16] at least for C non hyperelliptic, non trigonal, and non bielliptic (their hypotheses also exclude q =

$\iota(p)$ but their argument for irreducibility of "S_{pq}" does not require this; moreover we only need the result for one point $q \neq p$). This covers all our cases of double covers $\tilde{C}/C$ with $g(C) \geq 4$ and such that Ξ is smooth in codimension 3, except when C is non hyperelliptic of genus 4, hence trigonal. In this case we will check the hypotheses of Beauville's criterion in [Be] for $\mathcal{D}_p \cap \mathcal{D}_q$ to be reduced, normal, and irreducible hold for some $q \neq p$. Let $\pi: \tilde{C} \to C$ be the double cover and denote $\pi(p) = \overline{p}$ and $\pi(q) = \overline{q}$. We need to show for some q that $|\omega_C - \overline{p} - \overline{q}| \cong \mathbb{P}^1$ has no base points and gives a morphism with at most one ramification point in each fiber, at most of ramification index ≤ 3. The embedded canonical model of C in $\mathbb{P}^3$ projects with center $\overline{p}$ to a spanning plane curve of degree 5, hence a reduced plane quintic Δ with ≤ 2 singular points (since $\delta = p_a - g = 6-4 = 2$) and no point of multiplicity ≥ 3 (since C is non hyperelliptic). Choose any smooth point $\overline{q}$ on Δ not on any bitangent nor on any hyperflex (tangent line with intersection number ≤ 4) nor on any line through two singular points, nor on any tangent line through a singular point. Then the pencil of lines through $\overline{q}$ defines the base point free system $|\omega_C - \overline{p} - \overline{q}|$ and the corresponding morphism, which is projection from $\overline{q}$, satisfies Beauville's hypotheses. **QED for Claim 6.4.9.**

Claim 6.4.10. Both components of I_p meet $(1 \times \varphi)^{-1}(\tilde{C} \times U)$.
Proof: $\mathcal{D}_p$ meets $\varphi^{-1}(U)$ since $\mathcal{D}_p$ maps onto Ξ, and $\cup_{q \neq p} (\mathcal{D}_p \cap \mathcal{D}_q)$ is dense in $\mathcal{D}_p$ hence $\cup_{q \neq p} (\mathcal{D}_p \cap \mathcal{D}_q)$ also meets $\varphi^{-1}(U)$). **QED for Claim 6.4.10.**

By Claim 6.4.10, both components of I_p give components of S_p. Hence the components of I_p are exactly those of S_p, and have the same multiplicities, hence are also reduced.

QED for Lemma 6.4.7.

Remark 6.4.11. The previous argument shows that if $1\times\varphi:\tilde{C}\times\mathcal{D}_p\to\tilde{C}\times\Xi$ is the birational morphism induced by the restricted Abel map $\varphi_p:\mathcal{D}_p\to\Xi$, and I_p the restriction to $\tilde{C}\times\mathcal{D}_p$ of the incidence divisor, then $S_p = (1\times\varphi)(I_p)$. The method of producing Poincare divisors from incidence divisors is clearly explained in the most natural case of line bundles of degree g on the Jacobian of a curve of genus g in the notes of Kempf [Ke3, p.154], who attributes the idea to Riemann. If $Z\subset C^{(d)}$ is any reduced irreducible subvariety on which the Abel map $\alpha:Z\to W\subset \mathrm{Pic}^{(d)}$ is birational onto its image W, and such that W is normal and locally factorial (for example if $W = \mathrm{Pic}^d(C)$), and if I_Z is the restriction to $C\times Z$ of the incidence divisor $I \subset C\times C^{(d)}$, then the birational map $1\times\alpha:C\times Z\to C\times W$ carries I_Z to an effective Cartier divisor S on $C\times W$ which gives a line bundle whose restrictions to the fibers $C\times\{L\}$ for L in W, define the inclusion map $W\subset \mathrm{Pic}^{(d)}$. The case $Z = C^{(g)}$ and $W = \mathrm{Pic}^g(C)$, gives the unique Poincare line bundle which induces the map $C\to \mathrm{Pic}^{\vartheta}(\mathrm{Pic}^g(C))$ taking q to $\mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta_q)$, where $\vartheta = c_1(\mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta))$, and Θ_q is the image of Θ under the natural action of q in C on $\mathrm{Pic}^{g-1}(C)$, taking L to $L\otimes\mathcal{O}(q)$.
This is the unique effective Poincare line bundle $\mathcal{L}$ on $C\times \mathrm{Pic}^g(C)$ such that for q in C, $c_1(\mathcal{L}|\{q\}\times \mathrm{Pic}^g(C)) = \vartheta$, (in integral cohomology) [SV5].

The cases $Z_p = \mathcal{D}_p \subset C^{(g+1)}$ parametrized by points p of C, and W = $Pic^{g+1}(C)$, give the only effective Poincare line bundles $\mathcal{L}$ on $C \times Pic^{g+1}(C)$ such that for q in C, $c_1(\mathcal{L}|\{q\} \times Pic^g(C)) = \vartheta$. These Poincare bundles thus determine the curve C and hence give a version of a Torelli theorem [Ke1; Ke2, p.253, cor 4.4.c; cf. SV5, Prop.5.8]. The theorem in the present paper is an analog for Pryms of this result.

6.5 The effective Poincare bundle $\mathcal{L}_p = \mathcal{O}(S_p)$

Fix any point p in $\tilde{C}$, recall $X \supset \mathcal{D}_p = \{D \text{ in } X \subset \tilde{C}^{(2g-2)} : D \geq p\}$, and let $\gamma^*(\mathcal{D}_p) = S_p$ be the closure in $\tilde{C} \times \Xi$ of the preimage of $\mathcal{D}_p$ by the morphism defined by γ on its domain of regularity. To see S_p is a Cartier divisor in $\tilde{C} \times \Xi$, note that by hypothesis singΞ has codimension ≥ 4 in Ξ, so the singular locus of $\tilde{C} \times \Xi$ has codimension $\geq$ 4 in $\tilde{C} \times \Xi$. Since a local complete intersection which is locally factorial in codimension ≤ 3 is locally factorial by Grothendieck's proof of the "Samuel conjecture", $\tilde{C} \times \Xi$ is locally factorial [cf. BD], so any Weil divisor on $\tilde{C} \times \Xi$ is Cartier. Thus $\mathcal{L}_p = \mathcal{O}_{\tilde{C} \times \Xi}(S_p)$ is an effective line bundle on $\tilde{C} \times \Xi$. In fact $\mathcal{L}_p$ is a Poincare line bundle for $\tilde{C}$ in the sense that the map $L \mapsto \mathcal{L}_p|\tilde{C} \times \{L\}$ from Ξ to $Pic(\tilde{C})$ is the inclusion map $\Xi \subset Pic^{(2g-2)}(\tilde{C})$. To see this, first note that for L in U the restriction $S_p|\tilde{C} \times \{L\} = D \times \{L\}$, where D is the unique divisor of |L| containing p. This shows the morphism $L \mapsto \mathcal{L}_p|\tilde{C} \times \{L\} = \mathcal{O}(S_p|\tilde{C} \times \{L\}) \cong \mathcal{O}_{\tilde{C}}(D) = L$ is the inclusion map $\Xi \subset Pic^{(2g-2)}(\tilde{C})$, for L in U, hence for all L in Ξ by continuity. To see which Poincare bundle it is, let's look at the corresponding map $\tilde{C} \to Pic(\Xi)$ induced by $\mathcal{L}_p$. Let q be general in $\tilde{C}$ and consider the restriction of S_p to $\{q\}\times\Xi$. By the description above, this is precisely $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)\subset\Xi$. The following result that $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$ is due to Beauville and Debarre [BD].

Lemma 6.5.1. If C is non hyperelliptic of genus $g \geq 4$, and $\tilde{C}/C$ is a double cover with Ξ either smooth or smooth in codimension 3, then for any p on $\tilde{C}$, and any $q \neq p, p'$, if $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ has its reduced scheme structure, then $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$ as cycles and schemes, where $p' = \iota(p)$. Moreover if $g \geq 5$, or if $g = 4$ and q is general, then $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ is irreducible. [Note $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ is an improper intersection when q = p, and $\Xi\cdot\Xi_{a(q,p')}$ is improper when q = p'.]

Proof: By [BD, proof of Prop. 1, p.615], $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$ as sets, in particular the set $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ is pure dimensional. Then by [Be, Rmq. 1, p.360], $[\varphi(\mathcal{D}_p\cap\mathcal{D}_q)] = \varphi_*[\mathcal{D}_p\cap\mathcal{D}_q]$, and by [Be, Thm. 1, p.364], $\varphi_*[\mathcal{D}_p\cap\mathcal{D}_q] = \Xi\cdot\Xi_{a(q,p')}$ as cycles. Since Ξ is locally factorial $\Xi\cdot\Xi_{a(q,p')}$ and $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ are Cartier divisors which are equal as Weil divisors, hence since Ξ is normal they are equal as schemes [cf. Fu, Ex2.1.1,p.30, Ha, Prop.6.11, p.141]. If $g \geq 5$ then by [BD, p.615] for all p and all $q \neq p,p'$, $\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$ is reduced and irreducible in Ξ. If g = 4 this holds for all p and general q by the argument in Claim 6.4.9 above. **QED.**

Thus the induced map $\tilde{C}\to \mathrm{Pic}(\Xi)$ takes q to $\mathcal{O}_\Xi(\Xi_{a(q,p')})$, in particular the chern class of this line bundle is $\xi = c_1(\mathcal{O}_\Xi(\Xi))$ in

$H^2(\Xi,\mathbb{Z})$, and the map is a version of the Abel Prym map $a_p:\tilde{C}\to P_0$ via the isomorphism $P_0 \cong \mathrm{Pic}^{\xi}(\Xi)$. We will write $a_p':\tilde{C}\to \mathrm{Pic}^{\xi}(\Xi)$ also for this map, as well as other canonically equivalent incarnations such as the map $a_p':\tilde{C}\to \mathrm{Pic}^0(\Xi)$ taking q to $\mathcal{O}_\Xi(\Xi_{a(q,p')}-\Xi)$.

7. Proof that $NR(X) \cong NR(\mathcal{E}_p)$ for $\mathcal{E}_p = \mu_*(\mathcal{L}_p)$, $(\mu:\tilde{C}\times\Xi\to\Xi)$

Next we will push the Poincare line bundle $\mathcal{L}_p = \mathcal{O}_{\tilde{C}\times\Xi}(S_p)$ (defined in section 6.5) down to Ξ and relate it to X. As usual we assume $\tilde{C}/C$ is a double cover not on Mumford's list, i.e. C non hyperelliptic and Ξ either smooth or at least smooth in codimension 3.

The rank 2 "Prym sheaf" $\mathcal{E}_p$ on Ξ

Definition 7.1. Let $\mathcal{E}_p = \mu_*(\mathcal{L}_p)$, where $\mu:\tilde{C}\times\Xi\to\Xi$ is projection. Since the line bundles L in Ξ are generically pencils, $\mathcal{E}_p$ is by Grauert generically a rank 2 vector bundle on Ξ (e.g. it is a rank 2 vector bundle over Ξ_{sm}).

Proposition 7.2. Assume as always that $\pi:\tilde{C}\to C$ is a double cover with C non hyperelliptic of genus $g \geq 3$, and singΞ either smooth or smooth in codimension 3, and X, $\mathcal{E}_p$ are defined as above.

Then:

(i) $\mathbb{P}(\mathcal{E}_p|\Xi_{sm}) \cong X|\Xi_{sm}$.

(ii) $c_1(\mathcal{E}_p|\Xi_{sm}) = \xi|\Xi_{sm} = c_1(\mathcal{O}_{\Xi_{sm}}(\Xi))$ in $H^2(\Xi_{sm},\mathbb{Z})$.

(iii) $\mathcal{E}_p$ is "reflexive" (and coherent).

Proof of (i): This follows from Grauert's theorem [Ha, Cor.12.9, p.288-9], since over points L of Ξ_{sm}, the fiber of $\mathcal{E}_p$ is $H^0(\tilde{C},L)$ and the fiber of X is $|L| = \mathbb{P}H^0(\tilde{C},L)$. **QED.**

Proof of (ii): Begin by removing the singular locus of Ξ, and its inverse image in X and keep this convention throughout the argument, so that $\mathcal{E}_p$ is a rank 2 vector bundle and $c_1(\mathcal{E}_p)$ is defined. We will use the notation $c_{1,rat}(\mathcal{E}_p)$ to represent the chern class in $A^1(\Xi_{sm})$ of the vector bundle $\mathcal{E}_p|\Xi_{sm}$. Then the topological chern class $c_1(\mathcal{E}_p) = c_{1,top}(\mathcal{E}_p)$, is the image in $H^2(\Xi_{sm},\mathbb{Z})$ of the rational equivalence class $c_{1,rat}(\mathcal{E}_p)$ under the natural map $cl:A^1(\Xi_{sm})\to H^2(\Xi_{sm},\mathbb{Z})$ [Fu, ch. 19]. We will show $c_{1,rat}(\mathcal{E}_p) = [\Xi\cdot\Xi_{a(p,p')}]$ in $A^1(\Xi_{sm})$, from which it will follow that $c_{1,top}(\mathcal{E}_p) = cl([\Xi\cdot\Xi_{a(p,p')}]) = \xi$ in $H^2(\Xi_{sm},\mathbb{Z})$, as claimed. It suffices to check the stated inequality after restriction to $\mathcal{D}_p$. Indeed if $Z_p\subset\Xi$ is the image of the exceptional set of the restriction $\varphi_p:\mathcal{D}_p\to\Xi$ of the Abel map $\varphi:X\to\Xi$ to $\mathcal{D}_p$, we may restrict also to the complement of Z_p in Ξ_{sm}. I.e. consider the maps: $A^1(\Xi_{sm}) \to A^1(\Xi_{sm}-Z_p)\to A^1(\mathcal{D}_p-\varphi_p{}^{-1}(\text{sing}\Xi\cup Z_p))$. The first map is injective since it is restriction to the complement of the set $Z_p\cap\Xi_{sm}$ which has codimension two in Ξ_{sm}. The second map is induced by the isomorphism $\varphi_p:(\mathcal{D}_p-\varphi_p{}^{-1}(\text{sing}\Xi\cup Z_p))\to(\Xi_{sm}-Z_p)$ hence is also an isomorphism. Since the composition $A^1(\Xi_{sm}) \to A^1(\mathcal{D}_p-\varphi_p{}^{-1}(\text{sing}\Xi\cup Z_p))$ is thus injective, it suffices to check the equality $c_{1,rat}(\mathcal{E}_p) = [\Xi\cdot\Xi_{a(p,p')}]$ after pullback to $A^1(\mathcal{D}_p-\varphi_p{}^{-1}(\text{sing}\Xi\cup Z_p))$. I.e. it suffices to show that $[\Xi\cdot\Xi_{a(p,p')}]$ pulls back to $c_{1,rat}(\varphi_p{}^*(\mathcal{E}_p))$, via $\varphi_p{}^*$ restricted to $(\Xi_{sm}-Z_p)$. We will do this in two stages. If we denote $(\mathcal{D}_p-\varphi_p{}^{-1}(\text{sing}\Xi\cup Z_p))$ by $\tilde{\mathcal{D}}_p$, we show first that, on $\tilde{\mathcal{D}}_p$, $c_{1,rat}(\varphi_p{}^*(\mathcal{E}_p))$ equals $c_{1,rat}(\mathcal{N}(\tilde{\mathcal{D}}_p/X))$, the chern class of the normal bundle of $\tilde{\mathcal{D}}_p$ in X, and then we show that, again on $\tilde{\mathcal{D}}_p$, $c_{1,rat}(\mathcal{N}(\tilde{\mathcal{D}}_p/X)) = [\Xi\cdot\Xi_{a(p,p')}]$.

To compute the bundle $\varphi_p{}^*(\mathcal{E}_p))$ on $\tilde{\mathcal{D}}_p$, we represent it as an extension of line bundles via the fundamental exact sequence

$$(*)\ 0\to\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p\to\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p\to Q_p|\tilde{\mathcal{D}}_p\to 0,$$

where $\mathcal{O}_{\mathcal{E}_p}(-1)$ is the tautological line bundle on the projective bundle $\mathbb{P}(\mathcal{E}_p|\Xi_{sm}) = X-\varphi^{-1}(\mathrm{sing}\Xi)$, and where Q_p is the quotient line bundle to be determined. This lets us compute $c_1(\varphi_p{}^*(\mathcal{E})|\tilde{\mathcal{D}}_p)$ as the sum $c_1(\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p) + c_1(Q_p|\tilde{\mathcal{D}}_p)$. We claim that $\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$ is trivial and that $Q_p|\tilde{\mathcal{D}}_p \cong \mathcal{N}(\tilde{\mathcal{D}}_p/X)$. To trivialize $\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$ we find a natural non zero map from $\mathcal{O}_{\tilde{\mathcal{D}}_p}$ into $\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p$ with values in the sub linebundle $\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$. Recall that $\mathcal{E}_p = \mu_*(\mathcal{O}(S_p))$ where if $I_p\subset\tilde{C}\times\mathcal{D}_p$ is the incidence divisor consisting of pairs (q,D) such that q belongs to D in $\mathcal{D}_p$, then S_p is the image of I_p under the natural map $\tilde{C}\times\mathcal{D}_p\to\tilde{C}\times\Xi$ induced by $\varphi_p:\mathcal{D}_p\to\Xi$. Since this map is an isomorphism from $\tilde{C}\times\tilde{\mathcal{D}}_p$ to $\tilde{C}\times(\Xi_{sm}-Z_p)$, we may identify S_p with I_p on these open sets. Thus to give a section of $\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p =$ $\varphi_p{}^*(\mu_*(\mathcal{O}(I_p)))|\tilde{\mathcal{D}}_p$, we pull back by $\varphi_p{}^*$ a section s_p of $\mu_*(\mathcal{O}(I_p))$, i.e. a section s_p of $\mathcal{O}(I_p)$. We take of course a tautological section s_p, defining the incidence divisor I_p in $\tilde{C}\times\tilde{\mathcal{D}}_p$. Then at a point L in $\Xi_{sm}-Z_p$, the value of the section s_p in the fiber of $\mu_*(\mathcal{O}(I_p))$ at L is obtained by restricting s_p to the fiber $\tilde{C}\times\{L\}$. The fiber of the vector bundle $\mu_*(\mathcal{O}(I_p))$ at L is by Grauert the space $H^0(\tilde{C},L)$ and, if D is the unique divisor in $\tilde{\mathcal{D}}_p$ lying over L, then by construction the value determined by s_p in the fiber $H^0(\tilde{C},L)$ is an equation for the divisor cut by I_p on $\tilde{C}\times\{D\}$, i.e. it is an equation for D. Hence the value of the section $\varphi_p{}^*(s_p)$ in the fiber at D of $\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p$ does lie in the line determined by D in $H^0(\tilde{C},L)$, i.e. the value is in the tautological sub linebundle $\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$ of $\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p$. This section s_p thus defines a map $\mathcal{O}_{\tilde{\mathcal{D}}_p} \to \mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$ which we claim trivializes this bundle. I.e. since the section s_p defines the proper divisor D, it gives a non zero element of the fiber $H^0(\tilde{C},L)$, hence the associated section of $\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p$ is nowhere zero on $\tilde{\mathcal{D}}_p$.

Now we know $c_1(\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p) = 0$, so the fundamental exact sequence (*) above implies $c_1(\varphi_p{}^*(\mathcal{E}_p)|\tilde{\mathcal{D}}_p) = c_1(Q_p|\tilde{\mathcal{D}}_p)$. But $Q_p|\tilde{\mathcal{D}}_p \cong \mathcal{HOM}(\mathcal{O}_{\tilde{\mathcal{D}}_p}, Q_p|\tilde{\mathcal{D}}_p) \cong \mathcal{HOM}(\mathcal{O}_{\mathcal{E}_p}(-1)|\tilde{\mathcal{D}}_p, Q_p|\tilde{\mathcal{D}}_p) \cong \mathcal{T}_\varphi|\tilde{\mathcal{D}}_p$, the last equality by the standard representation of the tangent space to a Grassmanian at a given subspace, as the homomorphisms from the subspace into the quotient space, applied here to compute the tangent space to the projective space fiber of φ at a point of $\tilde{\mathcal{D}}_p$.

Next observe that $\mathcal{T}_\varphi|\tilde{\mathcal{D}}_p \cong \mathcal{N}(\tilde{\mathcal{D}}_p/X)$ = the normal bundle of $\tilde{\mathcal{D}}_p$ in X, since on $\tilde{\mathcal{D}}_p$ the normal space to the section equals the tangent space to the fiber. We will compute the chern class of this normal bundle in rational equivalence (which is just the linear equivalence class of the divisor of a section of the bundle), and show it corresponds to $[\Xi\cdot\Xi_{a_p'(p)}]$, via the isomorphism $\varphi_p:\tilde{\mathcal{D}}_p \to (\Xi_{sm}-Z_p)$.

Lemma 7.3. The isomorphism $\varphi_p:\tilde{\mathcal{D}}_p \to (\Xi_{sm}-Z_p)$ identifies $c_{1,rat}(\mathcal{N}(\tilde{\mathcal{D}}_p/X))$ with the restriction of $[\Xi\cdot\Xi_{a_p'(p)}]$ to $(\Xi_{sm}-Z_p)$.

Proof: If $Nm:X\to|\omega_C|$ is the restriction to X of the norm map on divisors $Nm:\tilde{C}^{(2g-2)}\to C^{(2g-2)}$, p,q are points of $\tilde{C}$ with images $\overline{p},\overline{q}$ on C, then the hyperplanes in $|\omega_C|$ dual to the points $\overline{p},\overline{q}$ on the canonical model of C pull back under Nm respectively to the divisors $\mathcal{D}_p+\mathcal{D}_{p'}$ and $\mathcal{D}_q+\mathcal{D}_{q'}$. In particular $\mathcal{D}_p+\mathcal{D}_{p'}$ and $\mathcal{D}_q+\mathcal{D}_{q'}$ are linearly equivalent on X, and hence also on $\tilde{\mathcal{D}}_p$. Thus we have

$\mathcal{N}(\tilde{\mathcal{D}}_p/X) \cong \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_p) \cong \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_p + \tilde{\mathcal{D}}_{p'} - \tilde{\mathcal{D}}_{p'}) \cong \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_q + \tilde{\mathcal{D}}_{q'} - \tilde{\mathcal{D}}_{p'})$. Now if $q \neq p,p'$, and $a(q,p)$ denotes $(1-\iota)(q-p) = (q-p) - (q'-p')$, then using the formula in [BD, p.615], $\mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_q) = \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_q \cap \tilde{\mathcal{D}}_p)$ = (in their notation $\mathcal{O}_{\tilde{\mathcal{D}}_p}(S_{pq})$), corresponds under the isomorphism $\varphi_p: \tilde{\mathcal{D}}_p \to (\Xi_{sm} - Z_p)$, to $\mathcal{O}_\Xi(\Xi_{a(q,p')})$, while $\mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_{q'})$ corresponds to $\mathcal{O}_\Xi(\Xi_{a(q',p')})$. Moreover by the formula proved in [SV1], $\mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_{p'})$ corresponds under this same isomorphism to (the restriction of) $\mathcal{O}_\Xi(\Xi)$. Thus we have $\mathcal{N}(\tilde{\mathcal{D}}_p/X) \cong \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_q + \tilde{\mathcal{D}}_{q'} - \tilde{\mathcal{D}}_{p'}) \cong$ (the restriction of) $\mathcal{O}_\Xi(\Xi_{a(q,p')} + \Xi_{a(q',p')} - \Xi) \cong$ (using the theorem of the square) $\mathcal{O}_\Xi(\Xi_{a(q,p') + a(q',p')}) = \mathcal{O}_\Xi(\Xi_{a(p,p')})$, since from the definition above $a(q,p') + a(q',p') = a(p,p')$.

Summing up, via the isomorphism $\varphi_p: \tilde{\mathcal{D}}_p \to (\Xi_{sm} - Z_p)$, we have $\mathcal{N}(\tilde{\mathcal{D}}_p/X) \cong \mathcal{O}_{\tilde{\mathcal{D}}_p}(\tilde{\mathcal{D}}_p) \cong \mathcal{O}_{(\Xi sm - Zp)}(\Xi_{a(p,p')})$. Since the rational chern class of a line bundle is simply the isomorphism class of the bundle, this proves what we want. **QED Lemma 7.3.**

Hence on $\tilde{\mathcal{D}}_p \cong (\Xi_{sm} - Z_p)$, we have $c_{1,rat}(\mathcal{E}_p) = c_{1,rat}(Q_p) = [\Xi.\Xi_{a_{p'}(p)}]$. Then, by injectivity of $A^1(\Xi_{sm}) \to A^1(\Xi_{sm} - Z_p)$, we have $c_{1,rat}(\mathcal{E}_p) = [\Xi.\Xi_{a_{p'}(p)}]$ also on Ξ_{sm}. This class is in fact not rationally equivalent to the Gauss class ξ, but it is cohomologous to it, so $c_{1,top}(\mathcal{E}_p) = cl([\Xi.\Xi_{a_{p'}(p)}]) = \xi$ on Ξ_{sm}, and we are done.

QED for (ii).

Proof of (iii): $\mathcal{E}_p$ is reflexive. Assume that C is nonhyperelliptic of genus $g \geq 3$ and let $\mathcal{L}$ be any Poincaré line bundle on $\tilde{C} \times \Xi$ (i.e. a line bundle $\mathcal{L}$ on $\tilde{C} \times \Xi$ such that $\mathcal{L}|_{\tilde{C} \times \{L\}} \cong L$ for each $L \in \Xi$), and let $\mathcal{E}$ denote the direct image $\mu_*(\mathcal{L})$ on Ξ, where $\mu: \tilde{C} \times \Xi \to \Xi$ is the projection to the 2nd factor. Recall that for $\mathcal{E}$ to be reflexive means that the natural map $\mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism. Let $\nu: \mathcal{E} \to \mathcal{E}^{**}$ denote the natural map of $\mathcal{O}_\Xi$-modules; we want to verify that ν is an isomorphism. First, ν is clearly an isomorphism over the open set $\Xi_{sm} \subset \Xi$ since there $\mathcal{E}$ is (the locally free $\mathcal{O}$-module corresponding to) a rank 2 vector bundle and vector bundles are reflexive.

To conclude the rest we will use the depth properties of $\mathcal{E}$ and $\mathcal{E}^{**}$. For any open set $V \subset \Xi$, let Z denote the closed set $\mathrm{Sing}(\Xi) \subset \Xi$ and consider the commutative diagram

$$\begin{array}{ccc} \Gamma(V,\mathcal{E}) & \xrightarrow{\nu_V} & \Gamma(V,\mathcal{E}^{**}) \\ \cap & & \cap \\ \Gamma(V-(Z\cap V),\mathcal{E}) & \to & \Gamma(V-(Z\cap V),\mathcal{E}^{**}). \end{array}$$

We already know that the bottom horizontal map (ν on sections over $V-(Z\cap V)$) is an isomorphism since $V-(Z\cap V) \subset \Xi_{sm}$, so it suffices to show that the 2 vertical maps are isomorphisms. For the one on the left, note that since $\mathcal{E} = \mu_*(\mathcal{L})$, the restriction map $\Gamma(V,\mathcal{E}) \to \Gamma(V-(Z\cap V),\mathcal{E})$ on Ξ is simply the restriction map $\Gamma(\mu^{-1}(V),\mathcal{L}) \to \Gamma(\mu^{-1}(V-(Z\cap V)),\mathcal{L})$ on $\tilde{C}\times\Xi$. Now $\tilde{C}\times\Xi$ is normal, $\mathcal{L}$ is a line bundle on $\tilde{C}\times\Xi$, and $\mu^{-1}(Z) = \tilde{C}\times Z \subset \tilde{C}\times\Xi$ is closed of codimension ≥ 2 (and of course $\mu^{-1}(V-(Z\cap V)) = \mu^{-1}(V)-(\mu^{-1}(Z)\cap\mu^{-1}(V))$, so this restriction is an isomorphism (by "Hartog's theorem"). For the one on the right, we apply "Schlessinger's lemma" [Sc] as follows. If Z is empty there is nothing

to do, so we may assume that $Z \neq \emptyset$ and then in particular (since C is not hyperelliptic) we have that Ξ is normal of dimension ≥ 2 and $\mathrm{codim}_{\Xi}(Z) \geq 2$. Therefore, since $\mathcal{E}^{**} = (\mathcal{E}^*)^*$, $\mathrm{depth}_Z(\mathcal{E}^{**}) \geq 2$ and hence $\Gamma(V,\mathcal{E}^{**}) \to \Gamma(V-(Z\cap V),\mathcal{E}^{**})$ is an isomorphism [SV3, Lemma 22 (with $\mathcal{G} = \mathcal{E}^*$), p. 392, and Prop. 18 ((i)$\Leftrightarrow$(iv), with n = 2), p. 391].
QED. for (iii), hence for Prop. 7.2.

Remark 7.4. Note, with regard to (i) in Proposition 7.2, that with $\mathcal{E}_p = \mu_*(\mathcal{L}_p)$ on Ξ we don't know a priori what vector space $\mathcal{E}_p|_L$ is for an arbitrary point $L \in \Xi$, i.e. at points where the hypotheses of Grauert's theorem do not hold. We would like to indicate a global description of X as the $\mathbb{P}^1$-fibration associated to a possibly different rank two $\mathcal{O}_{\Xi}$-module, which is valid also at those points. First, $X \subset \tilde{C}^{(2g-2)}$ is the Abel preimage of $\Xi \subset \mathrm{Pic}^{2g-2}(\tilde{C})$, so if $\varphi: X \to \Xi$ is the restriction of the Abel map to X, then for each $L \in \Xi$, $\varphi^{-1}(L) = |L| = \mathbb{P}(H^0(\tilde{C},L))$. Our convention is that, for a vector space V, $\mathbb{P}(V)$ denotes the space of lines in V through 0. Note that $\mathbb{P}(V)$ is then naturally $\mathrm{Proj}(S(V^*))$, where $S(V^*) = \mathbb{C} \oplus V^* \oplus S^2(V^*) \oplus \ldots$ is the symmetric algebra on V^* and is the graded algebra of regular functions on the vector space V. Thus we seek an $\mathcal{O}_{\Xi}$-module $\mathcal{F}$ such that for each $L \in \Xi$, the fibre $\mathcal{F}|_L$ is (naturally) isomorphic to the vector space $H^0(\tilde{C},L)^*$. Then $S(\mathcal{F})$, the symmetric algebra $\mathcal{O} \oplus \mathcal{F} \oplus S^2(\mathcal{F}) \oplus \ldots$ on $\mathcal{F}$, will have the property that $\mathrm{Proj}(S(\mathcal{F}))$ over Ξ has the same fibres as φ, i.e. $\mathrm{Proj}(S(\mathcal{F}))|_L = \mathrm{Proj}(S(\mathcal{F}|_L)) = \mathrm{Proj}(S(H^0(\tilde{C},L)^*)) = \mathbb{P}(H^0(\tilde{C},L)) = \varphi^{-1}(L)$. Thus, we expect that for such an $\mathcal{F}$, $\mathrm{Proj}(S(\mathcal{F}))$ is isomorphic to X (over Ξ).

Note that since $H^0(\tilde{C},L)^* \cong H^1(\tilde{C},\tilde{K}\otimes L^*)$ by Serre duality, in fact we want an $\mathcal{O}_\Xi$-module $\mathcal{F}$ such that $\mathcal{F}|_L \cong H^1(\tilde{C},\tilde{K}\otimes L^*)$, and this is easy to do. Namely, if $\mathcal{L}$ is any Poincaré line bundle on $\tilde{C}\times\Xi$, and if $f:\tilde{C}\times\Xi\to\tilde{C}$ and $\mu:\tilde{C}\times\Xi\to\Xi$ are the two projections, then take $\mathcal{L}' = f^*(\tilde{K})\otimes\mathcal{L}^*$ on $\tilde{C}\times\Xi$ and consider $\mathcal{F} = R^1\mu_*(\mathcal{L}')$. Then, since obviously $H^2(\mathcal{L}'|_{\tilde{C}\times\{L\}}) = 0$ for each $L \in \Xi$, by the "base change property" $\mathcal{F}$ is a coherent $\mathcal{O}_\Xi$-module such that $\mathcal{F}|_L \cong H^1(\mathcal{L}'|_{\tilde{C}\times\{L\}}) = H^1(\tilde{C},\tilde{K}\otimes L^*)$. Then for this $\mathcal{F}$ on Ξ, one can show that $\mathrm{Proj}(S(\mathcal{F})) \cong X$ (over Ξ).

8. Proof that $NR(\mathcal{E}_p) = a_p(\tilde{C})$ (= Abel Prym model of $\tilde{C}$)

Definition 8.1. The Abel Prym map $a_p:\tilde{C}\to P_0$ associated to a point p on $\tilde{C}$ is defined by $a_p(q) = a(q,p) = (1-\iota)(q-p) = (q-p)-(q'-p')$, where ι is the involution on $\tilde{C}$.

We show next that if $\mathcal{E}_p$ is the rank two sheaf defined in section 7, that $NR(\mathcal{E}_p) = \{\tau \text{ in } \mathrm{Pic}^0(\Xi): h^0(\mathcal{E}\otimes\tau) \neq 0\}$ can be naturally identified with the Abel Prym model $a_p(\tilde{C})$ of $\tilde{C}$ in P_0. It will then follow from Lemma 5.4 that $NR(X) = a_p(\tilde{C})$, up to translation in $\mathrm{Pic}^0(\Xi)$, and thus that X determines $\tilde{C}$. We will show below also how to recover the involution from $a_p(\tilde{C})$ and hence the double cover $\tilde{C}/C$ from X.

Proposition 8.2. $NR(\mathcal{E}_p) \cong a_p(\tilde{C})$ via the isomorphism $P_0\to\mathrm{Pic}^0(\Xi)$ taking a to $\mathcal{O}_\Xi(\Xi_a-\Xi)$.

Proof: The proof occupies most of the remainder of the paper. First we show we may reason about $\mathcal{L}_p$ instead of $\mathcal{E}_p$.

Lemma 8.3. $NR(\mathcal{E}_p) = \{\tau \text{ in } Pic^0(\Xi) : h^0(\tilde{C}\times\Xi, \mathcal{L}_p\otimes\mu^*(\tau)) \neq 0\}$.

Proof: I.e. $H^0(\tilde{C}\times\Xi, \mu^*(\tau)\otimes\mathcal{L}_p) \cong H^0(\Xi, \mu_*(\mu^*(\tau)\otimes\mathcal{L}_p)) \cong H^0(\Xi, \tau\otimes\mu_*(\mathcal{L}_p)) \cong H^0(\Xi, \tau\otimes\mathcal{E}_p)$, (by definition of μ_*, the projection formula, and the definition of $\mathcal{E}_p$). **QED.**

The usefulness of Lemma 8.3 is that $\mathcal{L}_p\otimes\mu^*(\tau)$ is a line bundle, hence is easier to compute with than the sheaf $\mathcal{E}_p$. Now we want to know which twists $\mathcal{L}_p\otimes\mu^*(\tau)$ by pull backs of degree zero line bundles τ from Ξ, are effective. $\mathcal{L}_p = \mathcal{O}(S_p)$ itself is effective since it is defined by the effective divisor S_p. Via the isomorphisms $P_0 \cong Pic^0(P) \cong Pic^0(\Xi)$ taking c to $\mathcal{O}_P(\Xi_c-\Xi)$ to $\mathcal{O}_\Xi(\Xi_c-\Xi)$, all degree zero line bundles on Ξ have form $\tau_c = \mathcal{O}_\Xi(\Xi_c-\Xi)$ for c in P_0.

Definition 8.4. For any point c in P_0, define $\mathcal{L}_c = \mathcal{L}_p\otimes\mu^*(\tau_c) = \mathcal{L}_p\otimes\mu^*(\mathcal{O}_\Xi(\Xi_c-\Xi)) \cong \mathcal{O}_{\tilde{C}\times\Xi}(S_p)\otimes\mu^*(\mathcal{O}_\Xi(\Xi_c-\Xi))$. In particular, $\mathcal{L}_p = \mathcal{L}_0 = \mathcal{O}_{\tilde{C}\times\Xi}(S_p)$.

We will show $\mathcal{L}_c$ is effective if and only if c belongs to $a_p(\tilde{C})$. Since $\mathcal{L}_p$ is a Poincare line bundle on $\tilde{C}\times\Xi$, so are all the $\mathcal{L}_c$. We will study the sections of the various line bundles $\mathcal{L}_c$ by representing them by not necessarily effective divisors and analyzing whether these divisors "move" by restricting them to the fibers of form $\{q\}\times\Xi$. By section 6.5, restricting $\mathcal{L}_p$ to the fibers $\{q\}\times\Xi$ determines the Abel Prym map $a_{p'}:\tilde{C}\to Pic^{\xi}(\Xi)$ taking q to $\mathcal{L}_p|_{\{q\}\times\Xi} \cong \mathcal{O}_\Xi(\Xi_{a(q,p')})$ for all q. In particular we are trying to determine all effective Poincare line bundles $\mathcal{L}$ on $\tilde{C}\times\Xi$ whose 1st chern class c_1 has Ξ component equal to ξ.

Lemma 8.5. $a_p(\tilde{C}) \subset NR(\mathcal{E}_p)$.

Proof: We must show for all q in $\tilde{C}$, that twisting $\mathcal{L}_p$ by $\mu^*(\tau_{a(q,p)})$ (= the line bundle corresponding to $a_p(q)$), is effective. Indeed we will show that $\mathcal{L}_p \otimes \mu^*(\tau_{a(q,p)}) = \mathcal{L}_q$. This will do it since $\mathcal{L}_q$ is effective. We will use the theorem of the square. Since by Lefschetz, the restriction map is an isomorphism $Pic^0(P) \to Pic^0(\Xi)$, the theorem of the square for $Pic^0(P)$ is also true in $Pic^0(\Xi)$.

Claim 8.6. $\mathcal{L}_q \otimes \mathcal{L}_p^* \cong \mu^*(\mathcal{O}_\Xi(\Xi_{a(q,p)} - \Xi))$.

Proof: We know if r is general, in particular different from p, p', q, q', then $S_p|\{r\} \times \Xi = \varphi(\mathcal{D}_p \cap \mathcal{D}_r) = \Xi . \Xi_{a(r,p')}$, and $S_q|\{r\} \times \Xi = \varphi(\mathcal{D}_q \cap \mathcal{D}_r) = \Xi . \Xi_{a(r,q')}$, whence $(S_q - S_p)|\{r\} \times \Xi = \Xi . (\Xi_{a(r,q')} - \Xi_{a(r,p')})$. Hence $\mathcal{L}_q \otimes \mathcal{L}_p^* = \mu^*(\mathcal{O}(\Xi_{a(r,q')} - \Xi_{a(r,p')}))$. Since $\mathcal{L}_q \otimes \mathcal{L}_p^* = \mu^*(\tau_c)$ for some unique τ_c in $Pic^0(\Xi)$, we must have $\mathcal{O}_\Xi(\Xi_{a(r,q')} - \Xi_{a(r,p')}) \cong \tau_c \cong \mathcal{O}_\Xi(\Xi_c - \Xi)$ for some unique c in P_0. Hence $\Xi_{a(r,q')} - \Xi_{a(r,p')}) \equiv \Xi_c - \Xi$ on P by Lefschetz, so by the theorem of the square we must have c = a(r,q') - a(r,p') = (r-q'-r'+q) - (r-p'-r'+p) = q-q'+p'-p = $a_p(q)$. **QED for Claim 8.6 and Lemma 8.5.**

Next we will show conversely, that $NR(\mathcal{E}_p) \subset a_p(\tilde{C})$.

By definition, $\mathcal{L}_c = \mathcal{L}_p \otimes \mu^*(\tau_c) = \mathcal{O}_{\tilde{C} \times \Xi}(S_p) \otimes \mu^*(\tau_c)$,

We want to use section 6.5 write these bundles entirely in terms of Ξ, and use the theorem of the square to simplify them. Since $\tau_c = \mathcal{O}_\Xi(\Xi_c - \Xi)$ is already defined in terms of Ξ, consider $\mathcal{O}_{\tilde{C} \times \Xi}(S_p)$.

Define Z_p = the divisor on $\tilde{C} \times \Xi$ defined by the pullback of the divisor $\tilde{C} \times \Xi$ on $\tilde{C} \times P$ via the map $\tilde{C} \times \Xi \to \tilde{C} \times P$ taking (q,L) to (q,L-a(q,p')).

Then as a set $Z_p = \cup_q \{q\}\times\Xi_{a(q,p')}$. By section 6.4 we know $S_p = \cup_q \{q\}\times\varphi(\mathcal{D}_p\cap\mathcal{D}_q)$. Since by section 6.5 $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$ for most q, S_p and Z_p are very similar, but they are not the same. I.e. when $q = p$, $\varphi(\mathcal{D}_p\cap\mathcal{D}_p) = \Xi$, while $\Xi\cdot\Xi_{a(p,p')}$ is a proper divisor on Ξ. On the other hand, when $q = p'$, $\varphi(\mathcal{D}_p\cap\mathcal{D}_{p'}) = \Gamma_{\overline{p}}$ is a Gauss divisor on Ξ [see SV1], while $a(p',p') = 0$ so $\Xi\cap\Xi_{a(p',p')} = \Xi$. Since $a(q,p) = 0$ if and only if $q = p'$, by comparison with S_p which we know has only two components, both reduced, it follows that Z_p has exactly two irreducible components, $\{p'\}\times\Xi$ and $M_p = \text{closure}(\cup_{q\neq p} \{q\}\times(\mathcal{D}_p\cap\mathcal{D}_q))$ $= \text{closure}(\cup_{q\neq p'} \{q\}\times(\Xi\cdot\Xi_{a(q,p')}))$. Thus Sp and Z_p each have two components, only one of which is shared. More precisely, $S_p = \{p\}\times\Xi \cup M_p$, while as a set, $Z_p = \{p'\}\times\Xi \cup M_p$.

Claim 8.7. The divisor Z_p on $\tilde{C}\times\Xi$, is reduced, so $Z_p = (\{p'\}\times\Xi) + M_p$.
Proof: We know the component M_p occurs simply in Z_p because when we project Z_p onto $\tilde{C}$ most fibers are scheme theoretically the same as those of the projection of S_p, namely the fiber over a general q is $\varphi(\mathcal{D}_p\cap\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$, equal as reduced schemes. (Since this shows S_p and Z_p have the same unique dominating component, it follows that in fact for all $q \neq p, p'$, that $\varphi_*(\mathcal{D}_p\cdot\mathcal{D}_q) = \Xi\cdot\Xi_{a(q,p')}$ is an equality of cycles.) It remains to show the component $\{p'\}\times\Xi$ is reduced in Z_p, which we do by calculating the order of vanishing of the function defining Z_p. If ξ is a Prym theta function defining Ξ on the universal cover of P, then by definition Z_p is defined locally by the function $\xi(q,t)$ where q is a variable on $\tilde{C}$ and t is a variable on Ξ. We may compute the multiplicity of the

component $\{p'\}\times\Xi$ at any general point, i.e. at (p',L) for L general on Ξ. This means we want to show the differential of $\xi(t-a_{p'}(q))$ does not vanish identically at $(q,t) = (p',L)$. But the partial differential in the $\tilde{C}$ variable $\partial\xi/\partial q$ is by the chain rule the differential of ξ evaluated on the tangent space of $\tilde{C}$ at p'. So it suffices to show that for general L, the kernel of $d\xi$ does not contain $T_{p'}\tilde{C}$, considered in the tangent space of P_0 at the origin, since $a_{p'}:\tilde{C}\to P_0$ is an embedding and $P \cong P_0$ by a translation inducing isomorphism of tangent spaces. Since for $q = p'$, we have $a_{p'}(q) = 0$, hence $L-a_{p'}(q) = L$, the kernel of $d\xi$ is the tangent space to Ξ at L. Since Ξ is irreducible, hence the Gauss map on Ξ is dominant, there is some L in Ξ_{sm} such that $T_L\Xi$ does not contain $T_{p'}\tilde{C}$. **QED for Claim 8.7.**

If we denote the projections by $f:\tilde{C}\times\Xi\to\tilde{C}$ and $\mu:\tilde{C}\times\Xi\to\Xi$, thus we have $\mathcal{L}_c = \mathcal{L}_p\otimes\mu^*(\tau_c) = \mathcal{O}_{\tilde{C}\times\Xi}(S_p)\otimes\mu^*(\tau_c) =$
$\mathcal{O}_{\tilde{C}\times\Xi}(Z_p + f^*(p-p'))\otimes\mu^*(\tau_c) = \mathcal{O}_{\tilde{C}\times\Xi}(Z_p + f^*(p-p') + \mu^*(\Xi_c - \Xi)) =$
$\mathcal{O}_{\tilde{C}\times\Xi}(\cup_q (\{q\}\times\{\Xi_{a(q,p')} + \Xi_c - \Xi\}) + f^*(p-p'))$. Now in analyzing how far this is from being effective, we would like to make the first part look as effective as possible. Using the theorem of the square, we have $\Xi_{a(q,p')} + \Xi_c - \Xi \equiv \Xi_{a(q,p')+c}$, so we replace
$\cup_q \{q\}\times\{ \Xi_{a(q,p')} + \Xi_c - \Xi\}+ f^*(p-p')$ by
$\cup_q \{q\}\times\{ \Xi_{a(q,p')+c} + f^*(p-p')$. Unfortunately this is not the same line bundle on $\tilde{C}\times\Xi$, in fact it is not even a Poincare line bundle for $\tilde{C}\times\Xi$. We have made the restrictions to the factors $\{q\}\times\Xi$ linearly equivalent to what they were, but we have changed the restrictions to the factors $\tilde{C}\times\{L\}$. We must thus correct by a pullback from $\tilde{C}$,

namely by $\cup_q \{q\}\times\{\Xi_{a(q,p')} + \Xi_c - \Xi - \Xi_{a(q,p')+c}\} \mid \tilde{C}\times\{L\}$ for general L in Ξ. But a general L does not belong to either Ξ_c or Ξ, so we must compute $\cup_q \{q\}\times\{\Xi_{a(q,p')} - \Xi_{a(q,p')+c}\} \cap \tilde{C}\times\{L\}$ =

$\{(q,L) : L$ belongs to $\Xi_{a(q,p')}\} - \{(q,L) : L$ belongs to $\Xi_{a(q,p')+c}\}$, and if we suppress the fixed L, this divisor on $\tilde{C}\times\{L\}$ is isomorphic to the following divisor on $\tilde{C}$,

$\cong \{q: -a(q,p')$ belongs to $\Xi_{-L}\} - \{q : -a(q,p')$ belongs to $\Xi_{c-L}\}$

$= \{q: a(q,p')$ belongs to $(-1)^*(\Xi_{-L})\} - \{q : a(q,p')$ belongs to $(-1)^*(\Xi_{c-L})\}$

$= a_{p'}{}^* \{(-1)^*(\Xi_{-L}) - (-1)^*(\Xi_{c-L})\}$

$= a_{p'}{}^* \{(\Xi_{c-L}) - (\Xi_{-L})\} = a_{p'}{}^* \{(\Xi_{c-L}) - (\Xi_{-L})\}$.

This is the divisor of degree zero on $\tilde{C}$ whose pullback to $\tilde{C}\times\Xi$ must be added to $\mathcal{O}_{\tilde{C}\times\Xi}(\cup_q \{q\}\times\{\Xi_{a(q,p')+c} + f^*(p-p')\})$ to make it equal the Poincare line bundle $\mathcal{L}_c$. We want to simplify this correction term a bit more. We claim in fact it is merely c, considered as a line bundle on $\tilde{C}$ via the inclusion $P_0 \subset \mathrm{Pic}^0(\tilde{C})$. First we need a standard isomorphism.

Lemma 8.8. If L is any point of $P \subset \mathrm{Pic}^{2g-2}(\tilde{C})$ the map sending $(\Xi_{c-L}) - (\Xi_{-L})$ to $(\Xi_c) - (\Xi)$ is an isomorphism $\mathrm{Pic}^0(\Xi) \cong \mathrm{Pic}^0(P_0)$, and (by the theorem of the square), every L induces the <u>same</u> isomorphism.

We assume Lemma 8.8 without proof.

Consequently we may as well write $(\Xi_c - \Xi) = \tau_c$ for the element $(\Xi_{c-L}) - (\Xi_{-L})$ of $\mathrm{Pic}^0(P_0)$. By Lemma 8.8 we can thus write the

correction term above as $a_{p'}{}^*\{(\Xi_c\text{-}L) - (\Xi\text{-}L)\} = a_{p'}{}^*(\tau_c)$. We will show below that the Abel Prym map $a_{p'}:\tilde{C}\to P_0$, induces by pullback of line bundles an isomorphism $\mathrm{Pic}^0(P_0) \to P_0 \subset \mathrm{Pic}^0(\tilde{C})$ which is inverse to the Prym polarization isomorphism
$P_0 \to \mathrm{Pic}^0(P_0)$. This will prove the correction term is just $a_{p'}{}^*(\tau_c) = c$, a point of P_0 which we will think of as either a divisor or a line bundle of degree zero on $\tilde{C}$.

Assume $a_{p'}{}^*(\tau_c) = c$, to be proved in Corollary 8.14 below.

Thus we have $\mathcal{L}_c = \mathcal{O}_{\tilde{C}\times\Xi}(\cup_q \{q\}\times\{ \Xi_{a(q,p')} + \Xi_c - \Xi\} + f^*(p-p'))$
$= \mathcal{O}_{\tilde{C}\times\Xi}(\cup_q \{q\}\times\{ \Xi_{a(q,p')+c} \}) \otimes f^*(c) \otimes f^*(\mathcal{O}(p-p'))$. Now denote $\cup_q \{q\}\times\{ \Xi_{a(q,p')+c} \}$ (restricted to $\tilde{C}\times\Xi$) $= Z_c$, an effective divisor on $\tilde{C}\times\Xi$. To see whether $\mathcal{L}_c = \mathcal{O}_{\tilde{C}\times\Xi}(Z_c)\otimes f^*(c)\otimes f^*(\mathcal{O}(p-p'))$ is effective, it is useful now to push it down to $\tilde{C}$ by the first projection $f:\tilde{C}\times\Xi\to\tilde{C}$. Note that $Z_0 = Z_p$ has the "vertical" component $\{p'\}\times\Xi$. However when c does not lie on the Abel Prym curve $a_p(\tilde{C}) = -a_{p'}(\tilde{C})$, then $a(q,p')+c$ is never zero for any q on $\tilde{C}$, hence then Z_c has no "vertical" component of form $\{q\}\times\Xi$. (Recall from section 2.1 that P_0 is the copy of the Prym variety in $\mathrm{Pic}^0(\tilde{C})$ that contains 0, while P_1 is the copy that does not.) Pushing $\mathcal{L}_c$ down by $f:\tilde{C}\times\Xi\to\tilde{C}$, we have $f_*(\mathcal{L}_c) = f_*(\mathcal{O}_{\tilde{C}\times\Xi}(Z_c))\otimes c\otimes\mathcal{O}_{\tilde{C}}(p-p')$, where c belongs to P_0 and $\mathcal{O}_{\tilde{C}}(p-p')$ belongs to P_1, hence the tensor product $c\otimes\mathcal{O}_{\tilde{C}}(p-p')$ belongs to P_1. Therefore it remains to determine $f_*(Z_c)$.

Lemma 8.9. Suppose Y is normal and irreducible, $f:Y\to B$ is a proper morphism onto a smooth curve B, and $\mathcal{L}$ a line bundle on Y such that $h^0(\mathcal{L}|f^{-1}(p)) = 1$ for p in a non empty open set U of B. If $\mathcal{L}$ =

$\mathcal{O}(D)$ where $D \geq 0$, and if for each point q_i on B the fiber $f^{-1}(q_i)$ appears in D with multiplicity m_i, then $f_*(\mathcal{L}) = \mathcal{O}_B(\Sigma m_i\, q_i)$.

Proof: By hypothesis, $D = E + \Sigma\, m_i\, f^{-1}(q_i)$, where E is an effective divisor on Y that does not contain any fibers of f. Thus by the projection formula, $f_*(\mathcal{O}(D)) = f_*(\mathcal{O}(E) \otimes f^*(\mathcal{O}_B(\Sigma m_i\, q_i)) = f_*\mathcal{O}(E) \otimes \mathcal{O}_B(\Sigma m_i\, q_i)$. Thus it suffices to show that $f_*\mathcal{O}(E) = \mathcal{O}_B$. First of all $f_*\mathcal{O}(E)$ is a line bundle on B, because it is a torsion free coherent sheaf on a smooth curve, and hence a vector bundle, and by Grauert's theorem the rank is $h^0(\mathcal{L}|f^{-1}(p)) = 1$ on the open subset U of B, hence the rank is one everywhere. The effective divisor E defines a section σ of $\mathcal{O}(E)$ hence a section s of $f_*(\mathcal{O}(E))$.

Claim: s is a nowhere vanishing section of $f_*(\mathcal{O}(E))$.

Proof of Claim: If s did vanish at p in Z, then in some open neighborhood V of p we could write $s = t\tilde{s}$ for some local parameter t at p and $\tilde{s}$ a local section of $f_*(\mathcal{O}(E))$. Then since $\Gamma(V, f_*(\mathcal{O}(E))) \cong \Gamma(f^{-1}(V), \mathcal{O}(E))$ is by definition an $\mathcal{O}_V$ module map, if $\tilde{s}$ corresponds to $\tilde{\sigma}$, the equation $s = t\tilde{s}$ in $\Gamma(V, f_*(\mathcal{O}(E)))$ yields the equation $\sigma = f^*(t)\tilde{\sigma}$ in $\Gamma(f^{-1}(V), \mathcal{O}(E))$, which would imply that σ vanishes along the fiber $f^{-1}(p)$, a contradiction.

QED for the Claim and Lemma 8.9.

Lemma 8.10.

(i) If c in P_0 does not lie on $a_p(\tilde{C})$, then $f_*(\mathcal{O}(Z_c)) = \mathcal{O}_{\tilde{C}}$, and $f_*(\mathcal{L}_c) = c \otimes \mathcal{O}(p-p')$ belongs to P_1.

(ii) If c does lie on $a_p(\tilde{C}) \subset P_0$, namely if $c = a_p(q)$, then $f_*(\mathcal{O}(Z_c)) = \mathcal{O}_{\tilde{C}}(q')$, and then $f_*(\mathcal{L}_c) = \mathcal{O}_{\tilde{C}}(q)$ in $\mathrm{Pic}^1(\tilde{C})$.

Proof: We only need to verify the hypotheses of the previous lemma, i.e. to compute orders of vanishing for our divisors on $\tilde{C}\times\Xi$.

Proof of (i): Since $Z_c = \cup_q \{ \Xi_{a(q,p')+c} \}$, if $a(q,p')+c \neq 0$ for all q, i.e. if $-c$ does not lie on $a_{p'}(\tilde{C})$, equivalently if c does not lie on $a_p(\tilde{C})$, then the intersection of the divisor Z_c with every fiber $f^{-1}(q) = \{q\}\times\Xi$ of f is a proper divisor $\Xi_{a(q,p')+c}$ in the fiber $\{q\}\times\Xi$. Thus the section defining Z_c does not vanish identically along any fiber, hence defines a section of $f_*(\mathcal{O}(Z_c))$ which does not vanish at any point of $\tilde{C}$, so that then $f_*(\mathcal{O}(Z_c)) = \mathcal{O}_{\tilde{C}}$. Then $f_*(\mathcal{L}_c) = c\otimes\mathcal{O}(p-p')$ belongs to $P_1\subset \mathrm{Pic}^0(\tilde{C})$, the component of $\mathrm{Nm}^{-1}(0)$ that does not contain $\mathcal{O}$, hence no line bundle in P_1 has sections. This proves that $h^0(f_*(\mathcal{L}_c)) = 0$ for c not in $a_p(\tilde{C})$, hence that $NR(\mathcal{E}_p)\subset a_p(\tilde{C})$.

Since we already know that $a_p(\tilde{C})\subset NR(\mathcal{E}_p)$, this will complete the proof of Prop. 8.2, modulo Corollary 8.14 below. We can look also at the other case profitably from the present point of view however.

Proof of (ii): If $c = a_p(q)$ does lie in $a_p(\tilde{C})$, then $-c = a_{p'}(q')$, and the divisor Z_c contains only the fiber $f^{-1}(q')$, which occurs with multiplicity one. This is true by the same $\partial\xi/\partial q$ calculation used above to prove Z_p is reduced along the component $\{p'\}\times\Xi$.

Then $f_*(\mathcal{O}(Z_c)) = \mathcal{O}_{\tilde{C}}(q')$, and since $a_p(q) = q-p-q'+p'$, hence $q' + a_p(q) + p - p' = q' + q - p - q' + p' + p - p' = q$, we have $f_*(\mathcal{L}_c) = \mathcal{O}(q')\otimes a_p(q)\otimes\mathcal{O}(p-p') = \mathcal{O}_{\tilde{C}}(q)$, the line bundle in $\mathrm{Pic}^1(\tilde{C})$ with one section vanishing simply at q.

QED for Lemma 8.10.

Corollary 8.11. For all p, the effective Poincare bundle $\mathcal{L}_p$ on $\tilde{C}\times\Xi$ has exactly one section.

Proof: By definition, $\mathcal{L}_c = \mathcal{L}_p \otimes \mu^*(\tau_c)$. Hence $\mathcal{L}_p = \mathcal{L}_0 = \mathcal{L}_c$ where $c = 0 = a_p(p)$ does lie on $a_p(\tilde{C})$. Hence by Lemma 8.10.(ii) $f_*(\mathcal{L}_p) = \mathcal{O}_{\tilde{C}}(p)$ has one section, as does $\mathcal{L}_p$. **QED.**

Remark 8.12. These examples show how a discontinuity can occur when we push down a moving divisor which specializes to contain a fiber. For instance here f_* of the general $\mathcal{O}(Z_c)$ is a trivial line bundle while f_* of the special $\mathcal{O}(Z_c)$ is a line bundle of degree one.

Abel Prym pullback is inverse to Prym polarization.

Finally we want to complete the proof of the formula $(a_p')^*(\tau_c) = c$, which was assumed just above Lemma 8.9 above. To finish, we need to know that pulling back line bundles via the Abel Prym map $(a_p')^*:\mathrm{Pic}^0(P_0)\to\mathrm{Pic}^0(\tilde{C})$, is precisely the composition of the inverse of the Prym polarization isomorphism $P_0\to\mathrm{Pic}^0(P_0)$, composed with the inclusion $P_0\subset\mathrm{Pic}^0(\tilde{C})$. Let there be given an étale connected double cover of smooth curves $\tilde{C}\to C$, with associated Jacobian and Prym varieties $P_0\subset\tilde{J}$, and principal polarizations $\lambda:J\to\mathrm{Pic}^0(J) = \hat{J}$, $\tilde{\lambda}:\tilde{J} = \mathrm{Pic}^0(\tilde{C})\to\mathrm{Pic}^0(\tilde{J})$, $\mu:P_0\to\mathrm{Pic}^0(P_0) = \hat{P}_0$. Denote by $\hat{j}$ the map dual to the inclusion $j:P_0\to\tilde{J}$, and by $h:\tilde{J}\to P_0$ the map such that $j\circ h = 1-\iota$ (i.e. $h = 1-\iota$ but with image P_0 instead of $\tilde{J}$), and that for any homomorphism f, f' denotes the dual homomorphism transferred back to the original ppav's, i.e. $f' = \text{polarization}^{-1}\circ\hat{f}\circ\text{polarization}$.

Lemma 8.13. The maps j and h form a "dual pair", i.e. $j' = h$, $h' = j$;

equivalently, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pic}^0(\tilde{J}) & \xrightarrow{\hat{j}} & \hat{P}_0 \\ \tilde{\lambda}\uparrow & & \uparrow\mu \\ \tilde{J} & \xrightarrow[h]{} & P_0 \end{array}$$

Proof: Since $\alpha: J\times P_0 \to \tilde{J}$ is an epimorphism, it suffices to show the two compositions $J\times P_0 \to \hat{P}_0$ given by $\hat{j}\circ\tilde{\lambda}\circ\alpha$ and $\mu\circ h\circ\alpha$ are equal. For this it suffices to check them equal on the two summands J and P_0 separately.

(0) Claim: on J, both compositions equal zero.

Since $\alpha: J\times P_0 \to \tilde{J}$, is given on J by $\pi^*: J = \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\tilde{C}) = \tilde{J}$, we want to look at $\mu\circ h\circ\pi^* = \mu\circ 0 = 0$, since π^* maps into the symmetric part of $\tilde{J}$, i.e. into the kernel of h.

On the other hand, $\hat{j}\circ\tilde{\lambda}\circ\pi^* = \hat{j}\circ\widehat{Nm}\circ\lambda$, using Mumford's result that Nm and π^* form a dual pair, $= (Nm\circ j)\hat{}\circ\lambda = \hat{0}\circ j = 0$, since $P_0 = \mathrm{im}(j) \subset \ker(Nm)$. QED for Claim.

(1) Claim: on P_0, both compositions equal 2μ.

Proof: On P_0, α is given by $j: P_0 \to \tilde{J}$. Then we have $\mu\circ h\circ j = \mu\circ 2p_0 = 2\mu$, since multiplication by 2 commutes with any homomorphism. Also we used the fact that $h\circ j$, the restriction of $1-\iota$ to P, is multiplication by 2. [Check it on the Abel Prym curve consisting of points of form $(1-\iota)(p-q) = p-q-p'+q'$. I.e. applying $1-\iota$ to this point gives $(1-\iota)(p-q-p'+q') = p-q-p'+q'-p'+q'+p-q = 2p-2q-2p'+2q' = 2(1-\iota)(p-q)$ as desired]. On the other hand, $\hat{j}\circ\tilde{\lambda}\circ j$ = by definition, the polarization induced by $\tilde{J}$ on P_0, which $= 2\mu$, i.e.

the polarization of $\tilde{J}$ restricts to twice the polarization of P_0, as is well known. **QED.**

Corollary 8.14. If $u:\tilde{C}\to P_0$ is an Abel Prym map, and $\hat{u}:Pic^0(P_0)\to J(\tilde{C})$ its induced dual map ("Abel Prym pullback"), then $\hat{u} \circ \mu = j$ (the inclusion $P_0 \subset \tilde{J}$), hence $\hat{u} = j \circ \mu^{-1}$.

Proof: If we define the Abel Prym map $u = h \circ t$, where t is the abel map $t: \tilde{C}\to \tilde{J}$, then on dual varieties we have $\hat{u} = \hat{t} \circ \hat{h}$. Hence if $\mu:P_0\to \hat{P}_0$ is the polarization of P_0, we have $\hat{u} \circ \mu = \hat{t} \circ \hat{h} \circ \mu$, and since we know the result on Jacobians, we have $\hat{t} = \lambda^{-1}$, where $\lambda:\tilde{J}\to Pic^0(\tilde{J})$ is the polarization of $\tilde{J}$, this yields $\hat{u} \circ \mu =$ $\hat{t} \circ \hat{h} \circ \mu = \lambda^{-1} \circ \hat{h} \circ \mu =$ (by definition) $h' = j$, by Lemma 8.13 above. **QED.**

Since in our earlier notation, we have $t = \tilde{\alpha}_{p'}$, and hence $u = h \circ t$ $= a_{p'}$, we get $(a_{p'})^* = \hat{u} = j \circ \mu^{-1}$, where μ is Prym polarization and j is inclusion $P_0 \subset \tilde{J}$. This completes also the proof of Prop. 8.2.

Recovering $\tilde{C}/C$ from NR(X).

Having recovered the curve $\tilde{C}$ from X, next we show how to recover also the involution $\iota:\tilde{C}\to \tilde{C}$, equivalently the double cover $\pi:\tilde{C}\to C$. We start from $NR(X) \subset Pic^0(\Xi)$ which corresponds under the isomorphism $Pic^0(\Xi) \cong P_0$, to $a_p(\tilde{C}) \subset P_0$, given up to translation.

Lemma 8.15. X determines not only the curve $\tilde{C}$, but also the involution $\iota:\tilde{C}\to \tilde{C}$, and hence the double cover $\pi:\tilde{C}\to C$.

Proof: If C has no $g^1{}_4$ so that the Prym canonical map $C\to \mathbb{P}^{g-2}$ is an embedding, then the Gauss map on $a_P(\tilde{C})$, which for non hyperelliptic C factors as the double cover $\pi:\tilde{C}\to C$ followed by the Prym canonical map, recovers π. In general, we use an argument of Welters [We2]. By definition, the projection $p = (1-\iota):\mathrm{Pic}^0(\tilde{C})\to P_0$, is determined by the involution ι on $\mathrm{Pic}^0(\tilde{C})$. Conversely, the involution ι on $\mathrm{Pic}^0(\tilde{C})$ is determined by the projection via $\iota = (1-p)$. Thus if $a:\tilde{C}\to P_0$ is (any translate of) the Abel Prym embedding of $\tilde{C}$ in P_0, since $\tilde{\alpha}:\tilde{C}\to \mathrm{Pic}^0(\tilde{C})$ is the Albanese variety of $\tilde{C}$, there is a unique map $p:\mathrm{Pic}^0(\tilde{C})\to P_0$ such that $a = p\circ\tilde{\alpha}$, i.e. p is the projection $\mathrm{Pic}^0(\tilde{C})\to P_0$. Then p determines the involution $\iota = (1-p):\mathrm{Pic}^0(\tilde{C})\to \mathrm{Pic}^0(\tilde{C})$ which by the strong Torelli theorem [Ma, p.792] determines also the involution $\iota:\tilde{C}\to\tilde{C}$. **QED.**

This concludes the proof of the main result Theorem 3.1.
The problem remaining open is to strengthen the analogy with Andreotti's theorem further and determine when $\tilde{C}\to C$ is determined by just the birational equivalence class of X. Conjecturally [cf. LS, SV5] this occurs when $\mathrm{Cliff}(C) \geq 3$.

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