On Classical Riemann Roch and Hirzebruch's generalization

Chapter 0. Summary of Contents

Mittag Leffler problems

We view the classical Riemann Roch theorem as a non planar version of the Mittag Leffler theorem, i.e. as a criterion for recognizing which configurations of principal parts can occur for a global meromorphic function on a compact Riemann surface. For example, the complex projective line $P^1 = C \cup \{\infty\}$, the one point compactification of the complex numbers, is a compact Riemann surface of genus zero. Any configuration of principal parts can occur for a global meromorphic function on P^1, and all such functions are just rational functions of z.

On a compact Riemann surface of genus one, i.e. a "torus", hence a quotient C/L of C by a lattice L, there is a residue condition that must be fulfilled. If f is a global meromorphic function on C/L and F is the L - periodic function on C obtained by composition C-->C/L, then the differential form F(z)dz must have residue sum zero in every period parallelogram for the lattice L. This necessary condition, properly stated, is also sufficient for existence of the function.

In the case of surfaces of genus g, such as the one point compactification of the plane curve $\{y^2 = f(x)$, where f is a square free complex polynomial of degree $2g+1\}$ there are g corresponding residue conditions. Note this surface may be viewed as a branched cover of degree 2 over P^1 by projecting on the x coordinate (and ∞ is always a branch point). Riemann's point of view was in the reverse order, since he considered as the basic object of study, a compact, connected branched cover of the projective line, and then proved such a manifold is a plane algebraic curve, possibly acquiring singular points from the plane mapping.

We discuss the classical proof of the theorem of Riemann and Roch, as well as the statement and some methods of proof for generalized versions in higher dimensions. We also present a few of the many applications of RR for curves and surfaces.

Riemann's approach

The problem Riemann set himself was, given a distinct set of points on a surface of genus g branched over P^1, to calculate the dimension of the space of meromorphic functions with poles only in that that set, and at worst simple poles. It is now stated more generally and more formally using the concept of "divisors", which are essentially zeroes and poles with multiplicities. A "divisor" is a finite formal linear combination of points, with integer coefficients $D = \sum n_i p_i$, on a smooth projective curve X. Then every meromorphic function f has an associated divisor div(f) consisting of the zeroes of f minus the poles of f, each counted with its appropriate multiplicity. The degree of a divisor $\sum n_i p_i$, is the sum of the integer coefficients, and for every meromorphic function f on a compact Riemann surface, the basic fact is that the degree of div(f) is zero, i.e. the number of poles and zeroes is the same, counted properly. For example a complex polynomial of degree n, has n finite zeroes, but also has a pole of degree n at ∞.

If L(D) is the space of meromorphic functions whose divisor bounded below by -D, i.e. such that $\text{div}(f) + D \geq 0$, in the sense that all coefficients of div(f) are $\geq$ the corresponding coefficients of -D, then the RR problem is to compute the analytic invariant dimL(D).

Riemann's approach was to solve first the analogous but simpler Mittag Leffler problem for meromorphic differentials, and then apply the usual criterion for exactness of differentials to pass to the case of functions. He always assumed for argument's sake, the distinctness of his points, i.e. the simplicity of his poles, but was well aware of the general case and remarked that it was easily dealt with by taking additional derivatives.

The cohomological approach

The modern approach to Riemann Roch for curves is to define analytic cohomology groups $H^0(D)$, $H^1(D)$, such that $H^0(D)$ and L(D) are isomorphic, and prove that "chi(X,D)" = $\dim H^0(D) - \dim H^1(D)$ [the analytic euler characteristic] is a topological invariant. Then the problem has two stages, **(i)** (Hirzebruch Riemann Roch or HRR) prove chi(D) = d+1-g, where d is the degree of D and g is the topological genus of X; this is essentially Riemann's theorem. **(ii)** (Serre Duality or SD) prove that $H^1(D)$ is isomorphic to the dual space $L(K-D)^*$, where K is the divisor of a differential form on X. This is Roch's contribution to Riemann's theorem. Together these give the classical RRT: dimL(D) - dimL(K-D) = d+1-g.

More precisely, if D is a positive divisor of distinct points, i.e. one with all coefficients equal to +1, Riemann essentially proved that L(D)/C is the kernel of a linear map from $C^d \dashrightarrow C^g$, hence $d-g \leq \dim L(D)-1 \leq d$, and also that dimL(K) = g. Roch later proved the codimension of the image of Riemann's map was dimL(K-D), hence that dimL(D)-1= d-(g - dimL(K-D)), where by Riemann's computation L(K) = g, then $\dim L(K-D) \leq g$.

The classical argument via residues

We give an account below of the classical proof of RR for curves, assuming Riemann's existence and uniqueness results for holomorphic and meromorphic differentials of second kind. The only other tool is classical residue calculus of differential forms, i.e. Stokes theorem. The proof in section III is the one originally given more briefly by Roch.

A recursive computation of the arithmetic genus of curves

After presenting Riemann's and Roch's arguments, we give a proof of stage **(i)** in the modern approach to the HRR for curves as follows: We use the simplest results of sheaf theory to prove that chi(D) - chi(O) = deg(D), by induction on deg(D). Then we prove chi(O) = 1-g, by immersing X in the plane as a curve of degree n with only nodes, and proving the formula for chi(O) by induction on n. The essential fact is that chi(O) depends only on the degree n, and can be defined for a reducible curve D+E with transverse components D, E , satisfying the inductive axiom chi(D+E) = chi(D)+chi(E)-chi(D.E), where chi(D.E) = deg(D.E). The formula 1-g has the same properties. This lets us go from knowing the equation chi(O) = 1-g for D and E separately, to deducing it for a smooth deformation of their union. The argument in section IV was inspired by the introduction of Fulton's paper [Am.J.Math, 101(1979)].

Most modern proofs of HRR in higher dimensions seem to be of this type: they establish axiomatic properties which characterize an invariant uniquely, then show that both chi(D) and an appropriate topological expression in the chern classes of D satisfy the axioms. The first such argument may be due to Washnitzer, although his published proof omitted part of the verification, later supplied by Fulton.

Arithmetic genus of smooth projective surfaces in P^3

To illustrate this method in higher dimensions we give a similar inductive argument in section IV for HRR on a smooth complex surface S which can be embedded in P^3. Here a divisor is a linear combination of irreducible curves on the surface. It is possible to define the intersection of a curve in D "with itself" obtaining at least an equivalence class of divisors on that curve, and a self intersectiion number, and then to do induction by appealing to the RRT on the curve.

Sheaf theory reduces the calculation of the difference chi(D) -chi(O) to the known HRR formula on the lower dimensional variety D, as follows: chi(D)-chi(O) = chi(O(D)$|_D$) = (D.D) + 1-g(D), where g(D) is the genus of the curve D. Then the adjunction formula relating the canonical divisors of S and D, gives 1-g(D) = -(1/2)[(K+D).D], hence chi(D)-chi(O) = (1/2)([D.(D-K)]. To get a formula for chi(D), it suffices to calculate chi(O) for the surface S.

Knowing in advance the answer to be chi(O) = (1/12)(K^2 + e(S)), where K is a "canonical" divisor of a differential 2 form, K^2 is the self intersection number of K, and e is the topological euler characteristic, we can check the validity of this equation for chi(O) by induction on the degree of the embedded surface S in P^3. Again chi(O) depends only on deg(S) = n, and satisfies the same inductive rule as before for transverse unions, chi(Y+Z) = chi(Y)+chi(Z)-chi(Y.Z). Since chi(P^2) = 1, we get chi(S_n,O) = chi(S_{n-1},O) + 1 - chi(C_{n-1},O) where C_{n-1} is a smooth plane curve of degree n-1. This determines chi(S_n,O) uniquely.

Then the adjunction formula for surfaces in P^3 and a Lefschetz pencil on S_n give (1/12)(e(S_n) + K^2) = n(n^2 - 6n + 11)/6. Checking that this formula satisfies the same inductive properties as does chi(O), gives the result, and hence the HRR for the smooth hypersurface S_n.

For stage **(ii)** of the modern proof of RRT for curves (the Serre duality result that $H^1(D)^*$ is isomorphic to L(K-D)), we sketch Serre's argument using the algebraic version of the classical residue pairing. He lumps the relevant spaces $H^1(D)^*$ for all D, into one large union, Weil's space of "adeles", which is infinite dimensional over the base field k, but one dimensional as a vector space over the field k(X) of meromorphic functions on the curve X. He then shows this union is isomorphic to the analogous one dimensional union of the spaces L(K-D) of meromorphic differentials for all D. It is then easy to check this isomorphism restricts for each D to the desired isomorphism $H^1(D)^*$ = L(K-D).

Kodaira vanishing theorems

In higher dimensions to go from HRR to a computation of dimL(D) even in special cases, requires more than Serre's duality. I.e. Serre duality only transforms $H^{\dim(X)}$ into an H^0, and we want our formula entirely in terms of H^0's when possible. Hence we appeal to Kodaira vanishing which gives a sufficient criterion for the higher cohomology groups H^1(D),....,$H^{\dim(X)}$(D) to vanish, and which holds for all sufficiently "positive" divisors. We state a modern version of this result apparently due to Rananujam, Mumford, Kawamata and Viehweg, but we do not give any proof.

The proof of the original vanishing theorem in the book of Kodaira and Morrow involves lengthy calculations in differential geometry and analysis to establish an inequality of "Bochner type" for curvature operators on harmonic forms on complex manifolds from which the result is easily deduced by Hodge theory. A short account of a more recent proof by Kolla'r can be found at the end of volume 3 of the introductory series by Ueno on algebraic geometry and scheme theory, (AMS translations). The Kodaira vanishing theorem is available only in characteristic zero, i.e. it is false in positive characteristic.

Computations using the Todd genus

Since the arguments we gave for HRR in dimensions 1 and 2 required knowing the topological formula for chi(O) in advance, we recall finally the formalism of Hirzebruch which shows in principle how to write down such formulas in all dimensions in terms of “chern roots”. The problem of giving the formulas in terms of chern classes is thus reduced to the elementary but tedious problem of expressing a given symmetric function in terms of elementary symmetric functions. Finally we compute the invariants of a cubic threefold, noting they agree with those of P^3, and discuss briefly how Clemens and Griffiths showed nonetheless a smooth cubic threefold is not even birational to P^3. Grothendieck’s relative version of the HRR for a map of two varieties is not discussed here; for this see Borel-Serre, or Fulton.

Caveat

The discussion below focuses on what RR says and its proof, not much on what it is good for. The RRT is a calculating tool par excellence, and has countless applications. A standard application is that the complex structure on a Riemann surface of genus g always occurs in exactly one of two ways, either as a double cover of P^1 with 2g+2 branch points, or as a smooth spanning curve of degree 2g-2 in P^{g-1}. As Griffiths says, the study of curves does not really get going until after the RRT is available. Almost the entire book by ACGH on geometry of algebraic curves assumes RRT for curves and other basic tools, and applies them extensively. Beauville’s beautiful book on algebraic surfaces shows how to classify surfaces not having too many holomorphic “multi” 2 forms, with RRT for curves and surfaces as a primary tool. Most people think using the RRT is more important than proving it, but for those curious about the statement and methods of proof, I hope the present discussion is useful. As a sample of its applications, there are a few exercises using RRT for curves at the end.

Acknowledgments: I am not an expert on Riemann Roch but I have used it for years on curves, and have always been fascinated by it. These notes are those of an amateur in the topic, intended for students or other interested learners. Any insights below come from reading or browsing first of all in Riemann’s works, then works by Siegel, Weyl, Gunning, Griffiths, Harris, Mumford, Walker, Serre, Fulton, MacDonald, Beauville, Kempf, Hirzebruch, Hartshorne, Atiyah, Bott- Tu, Washnitzer, and others, and from discussing the ideas with my friends and colleagues, especially Robert Varley.

Suggestions for reading

One of my favorite accounts of algebraic curves over the complex numbers, including a proof and applications of RRT, is Griffiths’ China lecture notes on algebraic curves. The statement of RRT as a residue criterion for existence of meromorphic functions, is very clear in Mumford’s yellow book on complex projective varieties and on pages 13-15 of Arbarello, Cornalba, Griffiths, Harris wonderful book on geometry of complex curves. The classical proof of Roch given below, is very clearly explained on pages 244-245 of Grifiths Harris.

At some point one should read Riemann's own account, as well as Roch's. There is now an English translation of Riemann's works from Kendrick Press, where the relevant arguments appear on pages 98-99. Roch's paper is in Crelle's Journal, vol 64 (1865) in very easy German. I do not recommend the English version in the volume of selections on Classical analysis, ed. by Garrett Birkhoff, from Harvard University Press. The articles are edited there rather unfortunately, essential remarks are removed, and misprints and errors are introduced so that Roch's argument is almost unintelligible, and Riemann's Brill - Noether estimate is transcribed incorrectly.

I recommend Fulton's paper on the arithmetic genus for the axioms characterizing chi(O), and his book Intersection Theory for a wealth of beautiful material including a short exposition of Grothendieck's version of HRR. Kempf's lectures on Abelian integrals from the University Autonoma in Mexico City are very clear in explaining the meaning and variational behavior of cohomology of line bundles on curves. I like Hirzebruch for sheaf cohomology on manifolds, although FAC is of course also outstanding. Kempf's little book on Algebraic Varieties is also excellent and amazingly succint for cohomology and a self contained proof of RRT and Serre duality (for vector bundles on curves). Beauville's book is a gem for Enriques' classification of surfaces. Serre's Groupes Algebrique... is a classic for curves in the abstract algebraic case, even irreducible singular ones, and Rick Miranda's book is a fine introduction to complex curves aimed at undergraduates which also adapts Serre's proof of RR. For traditional treatments that follow exactly the lines of Riemann's own work, but with the analysis and topology filled in, it is very instructive to consult Siegel's 3 volume work, and Weyl's classic The concept of a Riemann surface. Both these books have useful bibliographies including 19th century works.

Chapter I. Statement of RR for curves, and applications

Riemann's point of view was that functions are best understood by studying their largest domain of definition, a branched cover of the plane, and their singularities on that domain, hence meromorphic functions should be studied by examining their zeroes and poles on the full Riemann surface on which they exist. From this perspective, the fundamental existence question for meromorphic functions on a given Riemann surface is the "Mittag Leffler" problem: to determine which configurations of polar singularities, or of zeroes and poles, can arise from a global meromorphic function.

The Mittag Leffler problem in an open planar domain

If we restrict the domain of the functions to open subsets of the plane, as in the original Mittag Leffler problem, then we can assign polar behavior arbitrarily and always find a meromorphic function having this behavior. I.e. given any discrete set of points $\{p_i\}$, and any corresponding collection of polynomials P_i in $(z-p_i)^{-1}$ (without constant terms), there exists a globally defined meromorphic function f in the planar domain having as poles exactly the given set of points and having precisely the assigned polynomials as "principal parts" at those points, i.e. such that $f - \sum P_i$ is holomorphic in the entire plane.

The uniqueness question is much more complicated, since two functions with the same polar behavior differ by an entire function, and there is an enormous collection of entire functions defined on an open subset of the plane. In particular the space of functions with prescribed polar behavior is infinite dimensional.

The Mittag Leffler problem on a compact Riemann surface

When the Riemann surface under consideration is no longer planar, the global topology of the surface begins to play a role. On a compact (connected) surface the uniqueness problem is easy, since the only entire functions are constants, so the interesting question is existence, and now there can be certain types of polar behavior which cannot occur. Further, the fact that the space of entire functions on a compact surface is finite dimensional implies the space of meromorphic functions with given polar behavior is also finite dimensional, thus providing useful analytic invariants of the complex structure.

The "Riemann Roch problem" is that of computing the dimensions of spaces of meromorphic functions whose polar divisors are dominated by a given divisor. The classical solution, the "Riemann - Roch" theorem, expresses a simple relationship between the dimensions of spaces of meromorphic functions with bounded polar behavior, and topological invariants of the surface and the bounding pole divisor.

The Riemann Roch theorem

If a compact connected Riemann surface has genus g, then the "converse of the residue theorem" implies there are exactly g linear conditions on the infinite dimensional space of "principal parts", i.e. configurations of potential polar behavior, that must be satisfied for a configuration to occur as principal parts of a global meromorphic function. However these g conditions may not all be independent on a given finite dimensional subspace of principal parts.

For instance since P^1 has genus zero, there are no conditions, and every principal part occurs for the global rational function formed by adding up the Laurent polynomials defining the principal parts. Note that since $w = 1/z$ is a coordinate at $z = \infty$, hence at $w = 0$, the principal part $a/w + b/w^2 + ... + c/w^d$ at infinity is realized by the ordinary polynomial $az + bz^2 + ... + cz^d$.

On an elliptic curve, i.e. a Riemann surface X of genus one, there is one condition. E.g. if the elliptic curve X is expressed as the quotient C/L of the complex numbers C by a period lattice L, then a meromorphic function on X is equivalent to a doubly periodic function on C with the given lattice L as periods. Then the necessary and sufficient condition on a set of principal parts P is that the total residues of the differential Pdz in any one period parallelogram be equal to zero.

This generalizes to a compact connected Riemann surface X of any genus. If X has genus g, there are exactly g independent holomorphic differentials $w_1,...,w_g$ on X, and a set P of principal parts on X occurs for some meromorphic function if and only if each of the g locally defined differentials $Pw_1,...,Pw_g$ has zero total residue on X.

For example, given d general distinct points, we expect there to be at least d+1 - g independent meromorphic functions on X with at worst simple poles at these d points, (rather than d-g, since the constants always satisfy this condition). There may be more, since the g residue conditions may not be independent when restricted to a given d dimensional subspace of principal parts.

Riemann's inequality

If all g of the residue conditions are trivial on a given subspace of principal parts of dimension d, then there are d+1 such independent functions, the largest possible number. At the other extreme, if all g of the conditions remain independent then there are d+1- g such functions, the minimal possible number. This is Riemann's inequality: for any subset of d distinct points, the space of meromorphic functions having at most simple poles contained in that set has dimension between d+1- g and d+1.

Thus if we choose $d \geq g+1$ distinct points on X, there will always be some non constant meromorphic functions having at worst simple poles at those d points (and no poles elsewhere). A sequence of d+1 independent meromorphic functions with poles in a given set of d points define a map of C to a spanning curve of degree $\leq d$ in P^d. Since such a curve must have degree d, the map is of degree one, and since such a curve is isomorphic to P^1, thus C is isomorphic to P^1, and g = 0. Thus the upper bound d+1 only occurs for the curve P^1. More striking, whenever g = 0, the upper bound d+1 does occur, so every curve X of genus 0 is isomorphic to P^1. Hence a purely topological assumption yields an analytic conclusion - any Riemann surface homeomorphic to P^1 is also analytically isomorphic to P^1. Over a hundred years later, an analogous statement for surfaces has been proved. (A minimal complex algebraic surface diffeomorphic to P^2 is also isomorphic to P^2?)

Thus for curves of genus $g \geq 1$, Riemann's inequality says, if $L(p_1+...+p_d)$ is the space of meromorphic functions f with
$div_{pole}(f) \leq p_1+...+p_d$, then $d+1-g \leq \dim L(p_1+...+p_d) \leq d$.

Riemann's theorem for positive divisors on elliptic curves

In particular on a Riemann surface of genus g = 1, i.e. on an "elliptic curve", for any positive divisor $D = p_1+...+p_d$, where $d > 0$, we have the simple formula $\dim L(D) = \deg(D)$. I.e. Riemann's inequality is completely precise for positive divisors on elliptic curves, $d \leq \dim L(p_1+...+p_d) \leq d$.

Application to projective models of curves

A non constant meromorphic function on X with at most simple poles in the set $p_1,...,p_d$, defines a branched covering by X of P^1, with the inverse image of the point at infinity contained in the set $\{p_1,...,p_d\}$. If $d \geq g+1$, Riemann's inequality implies $\dim L(D) \geq 2$, so L(D) contains at least one non constant function. Thus every Riemann surface of genus g arises as a branched cover of P^1 of degree $\leq g+1$. To be sure Riemann assumed all his surfaces arose as branched covers of P^1, but made no hypothesis on the degree of the cover, so for him this is a "degree lowering" result for branched covers.

If g = 0, our curve is a degree one cover of P^1 and thus isomorphic to P^1. So there is only one isomorphism class of genus one curves.

If g = 1, every curve of genus 1 is a double cover of P^1, for instance by the map defined by the classical Weierstrass P - function.

If g = 1 and d = 3, we get two independent non constant meromorphic functions which embed our genus 1 curve in the projective plane as a cubic curve, as given classically by the P function and its derivative. The map must be an embedding since the image curve spans the plane, and the degree of the image curve times the degree of the map equals 3. So the map has degree 1 and the image curve has degree 3. Since a plane cubic has genus 1 if and only if it is smooth, the map is an embedding.

If g = 2, and d =3 we can represent the genus 2 curve as a 3 to 1 cover of P^1, but in this genus there is actually a 2 to 1 cover, provided we choose the 2 points carefully. To prove this we need Roch's enhancement of Riemann's theorem.

Roch's part of the theorem

It is easy to see a holomorphic differential w imposes no residue condition on the space of principal parts associated to d given distinct points if w vanishes at all the points. The converse is also true, and this refinement of Riemann's inequality is called the Riemann Roch theorem. E.g. when D is a sum of distinct points, the deviation (d+1) - dim(L(D)), of dim(L(D)) from the maximum possible value d+1, equals the dimension of the quotient space of all holomorphic differentials modulo the subspace of differentials vanishing on D.

A precise statement of the Riemann Roch theorem is as follows. Let D be any divisor on the compact connected Riemann surface X, i.e. let $D = n_1p_1+....+n_rp_r$ be a formal finite linear combination of points of X, with arbitrary integer coefficients n_i, and let $d = n_1+....+n_r$ be the "degree" of the divisor D.

Let div(f) be the divisor of a non zero meromorphic function f on S (divisor of zeroes minus divisor of poles, where each zero and pole is counted with its order), let L(D) be the space consisting of the zero function and of all non zero meromorphic functions f on S such that div(f) + D ≥ 0. Let K(-D) be the space of those meromorphic differentials w on S such that w = 0 or div(w) - D ≥ 0. Note that if D is positive, i.e. all coefficients $n_i > 0$, then L(D) consists of f such that f = 0 or the polar divisor of f is bounded above by D: $\mathrm{div}_{pole}(f) \leq D$, and K(-D) consists of holomorphic differentials whose zero divisor is bounded below by D: $\mathrm{div}_{zero}(w) \geq D$. As Mumford says, f is *allowed* to have poles on D and w is *required* to have zeroes on D.

Then we have the following statement:

Classical Riemann Roch formula [G.Roch, Jour. f. Math., 1865, vol.64]
dim(L(D)) = d + 1 - g + dim(K(-D)).

(Actually both Riemann and Roch only considered positive divisors it seems, but that is the main case.)

Easy consequences of Riemann Roch

In case D = K or D = 0, one of the integers, dim(K(-D)) or dim(L(D)), is readily computable. Then the theorem lets us compute the other one too, at least if we start with D = 0. I.e. if D = 0, then d = 0, and dimL(0) = 1 since the only holomorphic functions are constants. Thus we get that dim(K-0) = dimension of the space of holomorphic differentials = g. If D = (w) is the divisor of zeroes of a non zero holomorphic differential, then since the quotient of two holomorphic differentials is a meromorphic function with poles only where the denominator has zeroes, we see that dim(L(D)) = dim(K-0) = g, and dim(K-D) = dim(L(0)) = 1.

Then we recover an analytic antecedent of Hopf's theorem in smooth topology equating the index of a covector field with minus the euler characteristic, i.e. d = degree(w) = 2g-2. Thus a non zero holomorphic differential w has exactly 2g-2 zeroes counted with multiplicities. This implies whenever we have a divisor D with deg(D) = d > 2g-2, then dim(K-D) = 0, since L(E) = {0} whenever deg(E) < 0. Thus for all divisors D with deg(D) > deg(K), we have completely solved the problem of computing dimL(D), since for these divisors we have dim(L(D)) = d + 1 - g. (The Kodaira vanishing theorem stated in section 2 below is a generalization of this result to higher dimensions. I.e. if a divisor is "bigger" than the divisor of a differential in some sense, then all higher cohomology groups vanish.)

Now we can prove the branched covering result stated above for curves of genus 2. I.e. if $K = p_1+...+p_d$ is the divisor of zeroes of a non trivial holomorphic differential, then d = deg(K) = 2g-2 = 2, and dimL(K) = d+1-g + dimL(K-K) = 2+1-2+dimL(0) = 2. Thus the space L(K) contains a non constant meromorphic function on the curve having only two poles, i.e. a branched cover of P^1 of degree two. The difference between this argument and the one above using only Riemann's inequality, is that almost any three points can be the fiber of a 3 to 1 cover of P^1 by a genus 2 curve, but only a very special pair of points can be a fiber of a 2 to 1 cover of P^1. I.e. only a positive "canonical divisor" (the divisor of zeroes of a holomorphic differential) can be a fiber of a 2 to 1 cover, but any triple of points not containing a canonical divisor is a fiber of a 3 to 1 cover.

If g = 3, and D = K is a divisor of a holomorphic differential, then dimL(K) = 3 and we get a map f:C-->P^2 whose image X spans P^2 and such that the inverse image of the line at infinity is the divisor K of degree 4. Hence deg(f).deg(X) = 4, so X is either a smooth conic and f is a branched double cover, or else X is a plane quartic curve and f has degree one. Since a conic is isomorphic to P^1, this shows that curves of genus 3 come in two varieties, those which are branched double covers of P^1, and those which are degree one covers of plane quartics. Since a plane quartic has genus 3 if and only if it is smooth, the second case is actually an embedding onto a smooth plane quartic.

If we still assume the fact that any smooth plane quartic X has genus 3, it follows that the divisor D cut on it by a line L, has deg(D) = 4, and dimL(D) ≥ 3. Thus by Riemann Roch dimL(K-D) ≥ 1. Thus K ≥ D, and since both K and D have degree 4, then K = D. If z_0,z_1,z_2 are coordinates on P^2, and $\{z_0=0\}$ defines the line L, then the two meromorphic functions z_1/z_0 and z_2/z_0 that define the embedding of X in the plane belong to L(K). Thus the plane embedding is the one defined by the canonical divisor K.
I.e. all smooth plane quartics are canonically embedded Riemann surfaces of genus 3.

Now if there were also double cover f:C-->P^1, where C is a smooth plane quartic, then a fiber $E = p+q = f^{-1}(y)$ over a point would satisfy deg(E) = 2, and dimL(E) ≥ 2, (since both 1 and f belong to L(E)), hence we would have dimL(E) = d+1- g+dimL(K-E) = dimL(K-E). Since dimL(E) ≥ 2, then also dimL(K-E) ≥ 2, so there exist 2 independent holomorphic differentials w_1, w_2 both vanishing on the points p and q.

Then since there are altogether 3 independent holomorphic differentials on a curve of genus 3, let w_0 be a holomorphic differential not vanishing on p+q. Then the meromorphic functions $1 = w_0/w_0$, $f = w_1/w_0$, and $g = w_2/w_0$ form a basis of L(K), hence define the map of X into the plane assocated to the canonical divisor $K = \{w_0 = 0\}$. But this contradicts the fact that this map is an embedding, since that map takes p and q to the same point (1,0,0). Thus no smooth plane quartic can be a double cover of P^1.

Thus every curve of genus 3 is either a double cover of P^1 or a smooth plane quartic, and only one case can occur. Moreover a genus 3 curve can only be a double cover of P^1 in at most one way (up to automorphisms of P^1), since the pairs of points which can be fibers of such a double cover are determined as the fibers of the canonical map onto a conic in P^2.

This is only the beginning of the beautiful story of the geometric applications of RRT, for more of which see ACGH. Riemann deduced immediately from his proof of his theorem, that the smallest degree d such that every curve of genus g occurs as a branched cover of P^1 of degree d, is the smallest $d \geq (g/2) + 1$. We will recall how he derived this below. For now note that for g = 1,2, this gives d = 2, as we have seen above is true. For g = 3,4 we get d = 3, which can be realized for a plane quartic by projecting from a point of the curve, onto a line. A curve X of genus 4 which is not a double cover of P^1 embeds canonically as a curve of degree 6 in P^3, and so as to lie on a quadric surface. The quadric surface (at least if smooth) then projects along lines onto a plane conic, and this projection restricts to a degree 3 covering of P^1 by X.

The concept of an index

For the purpose of generalization, it is useful to regard the RR formula as an equality of "indices" as follows:

(*) dim(L(D)) - dim(K(-D)) = d + 1 - g.

In the formula above, the alternating sum on the left side, dimL(D) - dimK(-D) is the holomorphic Euler characteristic associated to the divisor D. The RR theorem then says the holomorphic Euler characteristic of any divisor is a certain explicit topological invariant.

Notice that the integer on the right side of (*), the topological index, does not depend on the complex structure of the surface S at all, but only on its topology, and that of the divisor. So although our real interest is in the integer dim(L(D)), the formula identifies a related integer, namely dim(L(D)) - dim(K(-D)), the analytical index, which is defined in terms of the complex structure, but which in fact does not depend on that structure. Since this latter integer is deformation invariant, it is much more computable than dim(L(D)), and essentially equates the problem of computing dimL(D) with that of computing dim(K(-D)).

There is a good analogy here with the problem of computing the number of vertices of a polyhedron. That number is not a topological invariant, but the alternating sum: vertices - edges + faces, i.e. V- E + F, is a topological invariant, namely the (topological) Euler characteristic. For a dodecahedron, this helps count the vertices as follows: the Euler characteristic is the same as for a tetrahedron, hence equals 4 - 6 + 4 = 2. There are 12 faces on a dodecahedron by definition, all pentagons, and each edge is shared by two faces, hence there are (5)(12)/2 = 30 edges, and thus we get v = e-f+2 = 30-12+2 = 20 vertices for a dodecahedron.

There is a cohomological interpretation in which L(D) and a cohomology space isomorphic to $K(-D)^*$ occur as kernel and cokernel of the same operator on an infinite dimensional space. Then the analytical index above is the index of this operator in the sense of functional analysis. A general theorem in functional analysis says the index of a Fredholm operator (a bounded operator with finite dimensional kernel and cokernel) is always deformation invariant, in the sense of being constant on connected components of the space of Fredholm operators. So RR is a precursor of such index theorems in analysis.

Chapter II. Statement and applications of RR for surfaces

For an algebraic surface S, we may analyze the RR problem along the same lines as described above for curves, by defining chi(D) in terms of cohomology groups, chi(D) = dim(L(D)) - $h^1(D)$ + $h^2(D)$, and ignoring for the moment the meaning of the numbers h^1 and h^2. This time D denotes a divisor on the surface S, i.e. a linear combination of irreducible curves on the surface.

1) Hirzebruch Riemann Roch: a topological formula for chi(D).
chi(D) = (1/12)(K^2 + e(S)) + (1/2)(D.[D-K]), where e(S) is the topological euler characteristic of S, and K^2 is the self intersection number of a "canonical" divisor K (the divisor of some meromorphic 2 form on S), and a "dot" means intersection number of divisors.

Steps of proof:

(i) (intersection numbers) A first step is to prove that chi(D) - chi(O) = (1/2)(D.[D-K]), a topological invariant where K is the divisor of any meromorphic 2 form. This is the easier part, and amounts to the "adjunction formula" relating the divisor of a differential on the curve D with the intersection of D with the divisor of a differential on the surface S. Unfortunately the numbers in the definition of the chis on the left hand side almost all remain mysterious. At least this shows the difference of any two chis is a simple topological invariant, an intersection number. This focuses attention next on understanding chi(O).

(ii) (Noether's formula) The second step is proving that chi(0) = (1/12)(K^2 + e(S)), where e(S) is the topological euler characteristic and K^2 is the self intersection of K, a "topological" invariant (of the analytic structure of S). The classical proof, i.e. Max Noether's own proof, uses a birational model of the surface S in P^3 where one can explicitly calculate both sides of the equation. This is harder than part (i), but it renders the mysterious integer chi(O) computable in terms of intersection numbers and the topological euler characteristic of the surface. In conjunction with part (i), this shows that chi(D) is always a topological invariant. I.e. not only is the difference of two chis topologically invariant, but in fact each chi is individually a topological invariant. We give an easy argument of this type below for a smooth complex surface in P^3.

Remark: Steps 1 and 2 together constitute what is called the Hirzebruch Riemann Roch theorem, a topological formula for chi(D). The key step is (ii): finding a topological formula for chi(O), the "arithmetic genus". To go further, given D we need to analyze the difference between the topologically computable invariant chi(D) and the truly analytic invariant dimL(D). There are two important results for doing this, i.e. for dealing with the extra terms $h^1(D)$ and $h^2(D)$.

2) Serre duality: This equates the mysterious integer $h^2(D)$ with the geometrically meaningful integer dim(K-D) = the number of holomorphic 2 forms on X which vanish on the curve D (assuming $D \geq 0$). Although we know nothing of h^1 except $h^1 \geq 0$, this already implies the following:

Classical RR inequality for a complex surface:
$\dim L(D) + \dim(K-D) \geq (1/2)(D.[D-K]) + (1/12)(K^2 + e(S))$,
where K is the divisor of a meromorphic 2 form and e(S) is the topological euler characteristic of S.

As before, this interpretation of h^2 yields a criterion for it to vanish.
If D is linearly equivalent to K+E where E >0 is a non zero effective divisor, then dim(K-D) = dim(-E) = 0, and hence:

Corollary: $\dim L(D) \geq (1/2)(D.[D-K]) + (1/12)(K^2 + e(S))$, for divisors of form D = K+E, where E > 0.

For example on a smooth quartic surface in P^3, K = 0, so this inequality holds for every non zero effective divisor D > 0. It also simplifies in that case (since intersecting with 0 gives zero), to the following: $\dim L(D) \geq (1/2)(D.D) + (1/12)(e(S)) = (1/2)(D.D) + 2$, by the computation of e(S) given below. Note however that D.D might be negative, even when D > 0!

Even when D is an arbitrary effective divisor on an arbitrary surface, where dim(K-D) may be non zero, Serre duality gives some information about the value of $h^2(D) = \dim(K-D)$, since elements of (K-D) are holomorphic 2 - forms that vanish on D, and we may be able to find such forms, or bound their dimension in a given example.

Serre duality also equates the terms $h^1(D) = h^1(K-D)$, but does little for understanding them, since both are equally mysterious. Thus it helps to have criteria for $h^1(D)$ to be zero, and the next result is often useful.

3) (strengthened) "Kodaira vanishing" This says dim(L(D)) = chi(D) when D = K+E, and E is not just effective but $E^2 > 0$, and $E.C \geq 0$ for all irreducible curves C on S. In this case both $h^2(D)$ and $h^1(D)$ are zero.

In higher dimensions, on a smooth projective variety X of dimension n over the complex numbers, $h^1(D), \ldots, h^n(D)$ are all zero if D = K+E where $E^n > 0$, and $E.C \geq 0$ for all irreducible curves C on X, e.g. E a hyperplane section or the inverse image of one under a generically finite morphism.

Remark: Kodaira's original version assumed E is the inverse image of a hyperplane section under a finite morphism, equivalently $E^n > 0$, and $E.C > 0$ for all curves C, instead of only $E.C \geq 0$. This slight weakening of the hypothesis, due apparently to Ramanujam, Mumford, Kawamata, Viehweg,..., seems to make the result significantly stronger.

Summary of RR for surfaces:
(a) (weak RR inequality for all divisors)
For any divisor D on a smooth projective complex surface S, we have: $\dim L(D) + \dim K(-D) \geq (1/2)(D.[D-K]) + (1/12)(K^2 + e(S)$.

(b) (strong RR inequality for divisors larger than K)
For a divisor D larger than a canonical divisor in the sense that D-K = E is effective, we have vanishing in the top degree $H^2(D) = K(-D)^* = 0$, and hence: $\dim L(D) \geq (1/2)(D.[D-K]) + (1/12)(K^2 + e(S))$.

(c) (precise RR equality for divisors much larger than K)
If D is much larger than K in the sense that D-K = E is "big" and "numerically effective" or "nef", in the sense that $E.E > 0$ and $E.C \geq 0$ for all irreducible curves C on S, e.g. if E is a hyperplane section, we have $H^1(D) = \{0\} = H^2(D)$, hence: $\dim L(D) = (1/2)(D.[D-K]) + (1/12)(K^2 + e(S))$.

Applications of HRR for surfaces:

Over the complex numbers, using Dolbeault cohomology and Hodge theory we can get more out of Noether's formula, since we can express chi(O) in terms of differential forms. I.e. $h^1(O)$ = dim(space of holo. 1- forms), and $h^2(O)$ = dim(space of holo. 2- forms), which interprets not only $h^2(O)$, but also $h^1(O)$ geometrically. Then
chi(O) = 1 - dim(space of holo. 1- forms) + dim(space of holo. 2- forms),
which is Hirzebruch's definition of chi(O).

As with compact curves, global holomorphic functions are constant, so the only <u>exact</u> holomorphic 1 form on a compact (connected) surface S is zero. Thus if S is also simply connected, e.g. any smooth surface in P^3, there are no non zero holomorphic 1 forms, so $h^1(O)$ = dim(space of holo 1- forms) = 0. This gives the formula for simply connected S: dim(2-forms) = $\dim L(K) = \text{chi}(O) - 1 = (1/12)(K^2+e(S)) - 1$, a useful special case of RR.

We have used the fact that the complex topological cohomology $H^1(S,C) = H^{1,0}.H^{0,1}$ is a direct sum of two mutually isomorphic subspaces, where $H^{1,0}$ = holomorphic one - forms, and $H^{0,1}$ is isomorphic to $H^1(O)$. Hence $h^1(O) = h^{0,1} = 0$ if and only if $h^{1,0} = 0$, if and only if $H^1(S,C) = 0$, as happens when S is simply connected.

Betti numbers of surfaces in P^3

On a smooth surface S of degree d ≥1 in P^3, a canonical divisor K is cut out by a hypersurface of degree d-4, and there are exactly as many independent 2 forms as homogeneous polynomials of degree d-4, i.e.
dim(K) = (1/6)(d-1)(d-2)(d-3). This also gives $K^2 = d(d-4)^2$. Thus:

chi(O) = 1 + dim(K) = 1 + (1/6)(d-1)(d-2)(d-3) = $(d/6)(d^2-6d+11)$.
Since by Noether chi(O) = $(1/12)(K^2 + e(S))$, and $K^2 = d(d-4)^2$, we get
e(S) = 12(1+dim(K)) - K^2 = $(2d)(d^2-6d+11) - d(d-4)^2 = d(d^2-4d + 6)$
= 2 + b_2.

E.g. on a smooth quartic surface in P^3, dim(K) = 1, and $b_2 = 22$. Thus there is essentially one 2-form, and twenty - two 2-cycles. I believe the vector of integrals of this form over an appropriate basis of these cycles determines the isomorphism class of the surface.

Line geometry on cubic surfaces

For a smooth cubic surface in P^3, we have K = - H, where H is a hyperplane section, so there are no 2 forms and no 1 forms, and Noether's formula says chi(0) = 1 = $(1/12)(K^2 + chi_{top}(S))$. Since -K is cut by a hyperplane, K^2 is the intersection number of S with a general line, i.e. K^2 = deg(S) = 3, so $chi_{top}(S) = 9$, and the second betti number $b_2 = 7$. We can draw some interesting geometric conclusions from topology as follows.

We know a general cubic surface contains a finite number of lines, and we choose one line m and project from it to P^1, sending each point p of S to the plane spanned by m and the point p, (if p lies on m, we map it to the tangent plane to S at p). This fibers S over P^1 with fibers which are plane sections of S residual to m, i.e. conics in S incident to m. If all these residual conics were non singular, i.e. if m did not meet any other lines on S, then by topology the Euler characteristic of S would be the product of the euler characteristics of the target P^1 and of the common fiber of the projection, a smooth conic also isomorphic to P^1. This would imply e(S) = 4. Since in fact e(S) = 9, there must be some singular conic fibers, i.e. either pairs of distinct lines, or doubled lines. If we assume that all singular conic fibers consist of pairs of distinct lines (which have Euler characteristic 3 instead of 2 for a smooth conic), it follows that each singular fiber contributes 3-2 = 1 more to e(S), than the product calculation of 4.

Since we need an excess contribution of 5 to get the right euler characteristic of S, there are exactly 9-4 = 5 pairs of lines meeting the line m on S. Thus every line on a cubic surface meets 10 other lines, in 5 pairs. In particular there is some line n on S not meeting m. Then the map f:S-->P^1 x P^1 given by the two projections from two given disjoint lines m, n sends a general point q of S to the pair of planes it spans, A = <q,m>, and B = <q,n> respectively. Given a pair of planes A,B, with A containing m, and B containing n, their line of intersection meets S at a point x of m, a point y of n, and another point q. The inverse image of such a pair of planes A,B under f, is the third intersection point q of S with the line A.B.

Hence the map f is an isomorphism except when the line A.B joining x and y, lies wholly in S. Since $b_2(S) = 7$, and $b_2(P^1 \times P^1) = 2$, and each line is a sphere generating a 2 - cycle, there are 5 such lines meeting both m and n on S. The morphism f collapses each of the 5 disjoint lines meeting both m and n on S to a different point of the smooth quadric $Q = P^1 \times P^1$, and is an isomorphism otherwise. Thus we can regard a smooth cubic surface as obtained by "blowing up" 5 distinct points on a quadric surface, replacing each point by a line.

If r is one of the 5 lines meeting both m and n, then there are two more lines s,t meeting respectively r and m, and r and n. It can be shown that if we project the smooth quadric Q to the plane P^2, by the rational map projecting from the point of Q to which r is collapsed, then the composition $f:S \to P^1 \times P^1 \to P^2$ is a morphism which collapses s and t instead of r, and is an isomorphism except for collapsing each of 6 disjoint lines on S to a different point of P^2. Thus S can also be considered as the result of blowing up 6 distinct points of P^2 to form lines of S. The resulting 6 lines on S, together with a single cubic curve on S cut by a plane section, can be taken as generators of the second homology group Z^7 of S. This also determines the intersection pairing on the group $H_2(Z)$.

One can enumerate all the lines on S, to show there are exactly 27 lines on a smooth cubic surface, and describe their configuration. This is explained nicely in Miles Reid's article in the Park City (PCMI) volume on complex algebraic geometry.

Chapter III. The classical proof of RRT for curves

Acknowledgments and references: Riemann's and Roch's arguments have been worked out in detail and clarified in more recent works, notably Weyl and Siegel, and they are also followed closely in the presentation by Griffiths and Harris. In fact, if one merely reads Riemann's division of differentials (or integrals) into first, second, and third kinds, and his mention of periods of integrals, it is not difficult to construct the argument below for Riemann's inequality. Roch's transition from Riemann's inequality to the full Riemann -Roch equation is then only a matter of using residue calculus to express Riemann's period map $W(D) \to C^g$ for meromorphic diferentials in terms of values of holomorphic differentials.

This argument of Roch follows directly from the reciprocity relations for meromorphic differentials, analogous to those proved by Riemann for holomorphic differentials. Roch's original paper is in Journal fur Math. (64),(1865), pp. 372-376. Riemann's proof is in his paper Theory of abelian functions, J. reine ang. Math. (54), (1857), in translation in: Bernhard Riemann, Collected Papers, Kendrick Press, 2004, tr. by R. Baker, C. Christenson and H. Orde, pages 94-99, especially page 99.

After discussing the existence of various standard meromorphic differentials, Riemann proves his part of the theorem in a few lines on page 99, followed by a brief derivation of what is now called the “Brill Noether” number for pencils on the same page, and begins the proof that every finite cover of the projective line is an algebraic plane curve, finishing on the next page. This amazing paragraph, section 5, of Abelian functions, appears on pages 102-109, of Riemann's reprinted Collected works in the Dover edition, 1953; and on pages 94-100 of the translation from Kendrick Press. I recommend very strongly reading at least page 99 of the Kendrick Press translation, as well as Roch's paper. Then one can follow up by reading pages 240-245, and 260-261 of [GH], for a fuller explanation of the arguments.

For expositions in the classical style, compare H. Weyl, The concept of a Rieman surface, 3rd edition, tr. by Gerald Maclane, 1955, pp. 135-137; and C.L. Siegel, Topics in Complex Analysis, vol. 2, Wiley - Interscience, 1971, pp.134-137.

Differentials on a Riemann surface

Riemann's approach to meromorphic functions on a Riemann surface was to analyze the differentials first (or rather their multivalued integrals), and use them to understand the (single valued) functions. The reason for this may be that the Mittag Leffler problem for differentials has a simpler answer than for functions. I.e. there is exactly one obstruction to the existence of a meromorphic differential with given principal parts: namely, each rational differential, regular on a punctured nbhd of a point q, has a well defined residue at q. A given configuration of principal parts at a finite set of points of a compact connected Riemann surface, i.e. a finite family of local rational differentials, occurs as the principal parts of a global meromorphic differential if and only if the residues add up to zero. Then to solve the ML problem for functions we just have to analyze which differentials are "exact", i.e. are of form df. This leads to the study of "periods" of differentials.

Natural operations on differential one forms:

1) The product of a meromorphic function f by a meromorphic differential w, is a meromorphic differential fw, and the quotient w/u of two non zero meromorphic differentials is a non zero meromorphic function.

2) The derivative df of a meromorphic function is a meromorphic differential with zero residue at every pole, and zero period around every loop.

3) The path integral of an arbitrary meromorphic differential is a meromorphic function which may be multiple valued (path dependent).

Remark: Given a configuration of principal parts for a meromorphic function, if we multiply these by a global differential w, we get a corresponding configuration of principal parts for a meromorphic differential. The resulting configuration arises from a global meromorphic differential v if and only if the original configuration arises from the global meromorphic function v/w. Since there are g independent global holomorphic differentials, they give in this way the g linear conditions which must be satisfied in order for a collection of principal parts to arise from a global meromorphic function. This explains the term (-g) on the right side of the RRT: dimL(D) - dimK(-D) = d+1-g.

From 2) above, we expect there are "more" differentials than (single valued) functions on a compact Riemann surface, and Riemann's point of view was to classify meromorphic differentials into three categories, and distinguish them for their different behavior with respect to differentiation and integration. He classified differentials (actually their integrals) as of1st kind, 2nd kind, and 3rd kind.

Differentials of first kind

A differential is of 1st kind if and only if it is holomorphic everywhere. Since two differentials with the same polar behavior differ by a holomorphic differential, understanding these is essential to the uniqueness part of the ML problem for differentials. Riemann proved the vector space of differentials of first kind has dimension g = the topological genus of the surface. Since there are no non constant holomorphic functions on a compact surface, if we integrate a non zero differential of first kind we get a holomorphic function which is never single valued.

Differentials of second kind

A meromorphic differential is of 2nd kind if it has zero residue at every singularity. The differential df of a meromorphic function f is always of this type. The converse is not true however, since differentials of 1st kind are also of 2nd kind, and as noted above the integral of such a differential may not be a single valued function. (Riemann himself considered only elementary differentials of second kind, with a double pole at one point. The present definition is that of Weyl.)

Differentials of third kind

A differential of 3rd kind is a meromorphic differential with at worst simple poles. As noted above, these never have single valued integrals, even locally around a pole. (Riemann considered the elementary examples having exactly two simple poles, with residues 1 and -1.)

With these definitions, a differential is of first kind if and only if it is of both second and third kind.

Existence of standard differentials

For each symplectic homology basis, [i.e. loops $A_1,\dots,A_g$, $B_1,\dots,B_g$ with all intersections trivial except for $A_j.B_j = 1 = -B_j.A_j$], Riemann's bilinear relations show that a holomorphic differential is determined by its A-periods, hence there can be at most a g dimensional space of them. He then argued, using the Dirichlet principle, that in fact the space of such differentials has dimension g. One can then construct standard differentials of first kind $w_1,\dots,w_g$, with integral of w_j equal to zero over every A loop except A_j, and integral = 1 over A_j.

There are also standard differentials of 2nd kind associated to each point as follows: for each point p on S, and any integer $n \geq 2$, there is a meromorphic differential on S of 2nd kind with pole only at p, of degree n, and with principal part of form dz/z^n. Using these standard ones, one obtains differentials of second kind with any desired polar behavior. Any two such with the same principal part at every pole must differ by a differential of 1st kind.

Riemann also constructed for each pair of points p,q, a standard differential of third kind with poles only at p, q, both simple, and with residues 1 and -1.

Converse of the residue theorem for differentials

Using these standard differentials of second and third kinds, one can deduce that given any finite set of poles and any set of principal parts at these points in terms of some local coordinates, a meromorphic differential exists having exactly these poles and these principal parts if and only if the residues at these poles add up to zero. Since any two meromorphic differentials with the same poles and principal parts differ by a holomorphic differential, by subtracting standard differentials of first kind, there is a unique meromorphic differential with given poles and principal parts (with zero total residue) which also has period zero around the A - cycles of a given symplectic homology basis.

Meromorphic functions as integrals of exact differentials

The differential df of a meromorphic function f is a meromorphic differential of second kind, and the principal parts of df faithfully reflect those of f, except the coefficient of z^{-n} in the principal part of f, occurs (after multiplication by -n) as the coefficient of z^{-n-1} in the principal part of df. In particular the coefficient of z^{-1} in df is always zero. Since a differential of second kind always exists with any desired principal parts at all, subject only to the requirement defining "second kind" that z^{-1} has coefficient zero, the existence question for meromorphic functions with given principal parts becomes the question of which differentials of 2nd kind are exact, i.e. which have form df.

A differential w of 2nd kind equals df for some single valued meromorphic function f, if and only if w can be integrated to give a single valued function, if and only if the path integrals of w are all path independent, if and only if w has integral zero around each closed loop on the Riemann surface. Since path integration of meromorphic differentials of second kind is a homology invariant, it is sufficient to check this on a basis of 2g independent homology cycles. Thus there are 2g linear conditions on the space of differentials of 2nd kind, and the kernel of this map is the space of "exact differentials", i.e. the space of all differentials of form df for global meromorphic functions f.

Thus, Riemann's approach to the analysis of global meromorphic functions on S is via the space of differentials of 2nd kind, and the analysis of their "periods". This leads immediately to Riemann's inequality as follows.

Riemann's proof of his theorem.

We restrict as Riemann did, to the case of a generic effective divisor consisting of distinct points. Let $D = \{p_1,.....,p_n\}$ be any finite collection of distinct points on S, and let L(D) be the vector space of meromorphic functions on S with poles at most supported in D and of order at most 1 at each point. We want to estimate the dimension of L(D) using Riemann's point of view.

Consider the space W(D) of meromorphic differentials with poles supported in D, of order at most 2 at each point, and of zero residue at each point and having zero periods around each A cycle for some fixed symplectic homology basis. (This normalization is the one used by Roch). Then W(D) has as basis, one of Riemann's standard differentials of 2nd kind for each of the n points p_i, so dim(W(D)) = n.

Differentiation is a linear map d:L(D)-->W(D), with one dimensional kernel, and image isomorphic to L(D)/C, so we would like to estimate the dimension of its image, the subspace of exact differentials in W(D). Consider the linear map W(D)-->C^g, given by integration over the B - cycles of our symplectic homology basis, with values (the "B-periods") in the space C^g. The exact differentials are precisely the kernel of this map, i.e. those with zero periods. Thus we have an exact sequence:

0-->C-->L(D)-->W(D)-->C^g.

The kernel L(D)/C of W(D)-->C^g thus has dimension $\geq$ n - g. On the other hand, dim(L(D)/C) $\leq$ dim(W(D)) = n. Since dim(L(D)/C) = dimL(D) - 1, this gives the inequality $n-g \leq \dim(L(D))-1 \leq n$.

Riemann's inequality: $\deg(D)+1-g \leq \dim(L(D)) \leq \deg(D)+1$, if D is effective. (Our argument assumed that D consists of distinct points, but this is inessential.)

Remarks: (i) We do not need to make the A periods of the differentials zero for this argument, since if we remove that requirement in the definition of W(D), then $\dim W(D) = n+g$, and the period map goes into C^{2g}, so the kernel L(D)/C still has dimension at least $n+g-(2g) = n-g$. Moreover the only exact holomorphic differential is zero as we said, so the kernel L(D)/C misses the g dimensional subspace of holomorphic differentials, so $\dim(V) \leq n$ as before. This is Riemann's original argument..

(ii) As observed above, the upper bound in Riemann's inequality is only reached for the surface P^1. I.e. if $\deg(D) = n$, then n+1 independent functions in L(D) give an embedding of S isomorphically onto a rational normal curve of degree n in P^n, which implies S is isomorphic to P^1. Thus we have $\dim L(D) = n+1$ if and only if $g = 0$, so if $g \geq 1$, $D>0$, $\deg(D)+1 - g \leq \dim(L(D)) \leq \deg(D)$. This is precise for $g = 1$.

Cor: Riemann Roch for effective divisors on elliptic curves:
$\dim(L(D)) = \deg(D)$, when $D > 0$, and $g = 1$.

(iii) Torelli's theorem says that given any symplectic homology basis, $A_1,\dots,A_g$, $B_1,\dots,B_g$ on a curve X, if the standard differentials of first kind $w_1,\dots,w_g$, have integral of w_j equal to zero over every A loop except A_j, and integral 1 over A_j, then the (g by g) matrix of integrals of the w's over the B periods uniquely determines the curve X up to isomorphism.

Remarks on the existence of differentials

From this perspective, the whole burden of proof of Riemann's result is on existence of differentials of 1st and 2nd kinds. For this, he used the Dirichlet principle (justified later by Neumann and Hilbert), Griffiths - Harris apply the Kodaira vanishing theorem for complex manifolds, and for plane curves one can just write down differential forms using adjoint polynomials, as Riemann himself observed.

For example on the plane cubic curve E: $\{y^2 = (x-a)(x-b)(x-c) = f(x)\}$, where a,b,c are distinct, by taking d of both sides we have $2y\,dy = f'(x)dx$, so $dx/y = 2dy/f'(x)$. If we note that at every point either y or f'(x) is non zero, we see that dx/y is a holomorphic differential, and this can also be checked at infinity.

On a smooth projective plane curve X of degree d, with affine equation $F(x,y) = 0$, the differential $dx/(\partial F/\partial y)$ is holomorphic on X and remains so when multiplied by any polynomial of degree $\leq d-3$. It follows that the number of independent holomorphic differentials on X is at least equal to the number of independent such polynomials, i.e. at least $(1/2)(d-1)(d-2)$.

This number $(1/2)(d-1)(d-2)$ is shown below to equal the topological genus g of the projective plane curve X, so this exhibits at least g independent holomorphic differentials. There cannot be more than g holomorphic differentials on X, since the direct sum of the holomorphic plus antiholomorphic differentials injects (via path integration) into the dual of the first singular homology space of the curve, which has dimension 2g. This map is injective because no harmonic form can be exact, by the maximum principle for harmonic functions. Thus the number of independent holomorphic differentials on a smooth plane curve X of degree d, is exactly g = (1/2)(d-1)(d-2).

Riemann gives this description of the holomorphic differentials on a plane curve and remarks that one could also write down the meromorphic differentials equally well, but he declines to do so explicitly, since he says the principle is clear. Note that if these two types of differentials are available, then the Riemann Roch theorem is completely proved for plane curves. Hence in spite of questions about Riemann's use of the so called Dirichlet principle to produce such differenrials in general, his proof of his theorem in the case of plane curves seems not to suffer from such doubts. Thus the work of Brill and Noether, giving a purely algebraic proof of RRT, while valuable as a way of rendering the theorem into algebra, and thus generalizing the coefficient field, seems unnecessary as a way of bolstering the foundations of the result over the complex numbers.

Riemann Roch on P^1

On the Riemann sphere, the surface of genus zero, there are no restrictions for principal parts of functions. Given any finite set of points and Laurent polynomials centered at them, the sum of these polynomials is itself a meromorphic function with the desired behavior.

Riemann Roch on a complex curve of genus one

Recall that the universal covering space of a compact Riemann surface must be a simply connected Riemann surface, hence by Riemann's classification theorem, must be either the sphere, the unit disc, or the plane. The sphere cannot map as a covering space of a surface of genus one since such a map cannot raise genus by Hurwitz' theorem. Since the disc has a metric of constant negative curvature, it cannot cover a surface of euler characteristic zero by the Gauss Bonnet theorem, which leaves only the complex plane. Thus a Riemann surface of genus one is a quotient of the complex plane by a discrete group of automorphisms isomorphic to Z^2, which one can no doubt show without great difficulty. is a lattice, i.e. generated by two real independent translations. If we show how to construct standard differentials of first and second kind on the quotient X of the complex plane by a lattice, we thus prove the RR theorem in genus one.

The differential dz is a global holomorphic differential of the first kind on X. It is usual to represent meromorphic functions on X as doubly periodic functions in the plane with respect to the lattice, and if P(z) is the famous Weierstrass P function, with a double pole and zero residue at every lattice point, and otherwise holomorphic, then P(z)dz is a standard differential of second kind corresponding to the point 0. Translating this construction to other points, yields all standard differentials of second kind and thus proves the RRT for our genus one surface X, i.e. dimL(D) = deg(D), at least in the case where D is a positive divisor consisting of distinct points.

Remark:

To try to prove RR this way for Riemann surfaces of higher genus, we could represent them as quotients of the unit disc by the action of discrete groups of automorphisms, and try to write down Eisenstein series for the various differential forms we need.

Roch's part of the RRT for curves of genus g

Next we want to derive the full RRT theorem, i.e. the formula
dimL(D) = deg(D) +1 - g + dimK(-D), at least for a positive divisor D consisting of distinct

points, and assuming the existence of standard differentials of first and second kinds.

We extend the previous discussion as in [Griffiths - Harris' Principles..., pages 244-245; C.L. Siegel, Topics in complex analysis, vol. 2 chap 4, and H. Weyl, The idea of a Riemann surface], using the classical bilinear relations for periods of integrals [pages 240-241 in G-H]. The discussion in G-H is very clear, but having learned intersection theory from Weyl, I prefer to do the proofs without dissecting the Riemann surface. This reciprocity argument is Roch's original one, but he omits the details, citing the similarity to Riemann's argument in the holomorphic case.

One uses Green's theorem to give relations among the path integrals of differential forms of first and second kinds. In a nutshell, we construct the Poincare duality isomorphism directly for a smooth compact Riemann surface X, constructing smooth differential forms associated to loops on S, so that if ω and η are Poincare dual to loops A and B, respectively, then A.B = $\int_B \omega$ = $-\int_A \eta = \iint_X \omega \wedge \eta$.

Briefly one does it for a symplectic basis of oriented loops on X. Given a loop A_i, one forms a narrow collar around it diffeomorphic to
S^1x I, and constructs a smooth function f that is constant on circles and grows along I from 0 to 1, then smooths it out to be constantly 0 at one end and constantly 1 at the other. Then df is a smooth form which can be extended by zero to all of X, and whose integral equals one over any path passing simply from the boundary S^1 x {0} of the collar to the other boundary S^1 x {1}.

In particular, integration of df over B_i equals 1, but any other basic cycle has a representative not meeting the support of the form, hence those integrals are zero. Thus integration of a loop γ against df equals intersection of γ with A_i, since that is true when tested on every basic homology cycle γ. Taking linear combinations of the forms associated to the basic cycles, defines a form ω associated to any cycle A, and such that A.B = $\int_B \omega$ for every cycle B.

If η is the form associated to B, then the equation $\int_B \omega = \iint_X \omega \wedge \eta$ follows from Green's theorem. We need only prove it for the two basic cycles associated to A_i and B_i, since these are the only ones whose forms have supports with non empty intersection. Their supports, i.e. the small collars around the cycles, meet in a small rectangle where one applies Green's theorem as in [p. 361ff., Lectures on Riemann Surfaces, World Scientific Publishing, 1989, ed. Cornalba, Gomez Mont, Verjovsky].

Riemann's bilinear relations for holomorphic forms

Since the intersection pairing on $H_1(X,C)$ is symplectic, if A_i, B_j is a symplectic homology basis, and $\sigma = \sum a_iA_i + \sum b_jB_j$, and
$\tau = \sum a'_iA_i + \sum b'_jB_j$, are two oriented loops, then $\sigma.\tau = \sum a_ib'_i - \sum a'_ib_i$.
If these loops σ and τ are dual to the forms ω and η (in the sense above that integrating against the form is the same as intersecting with the dual loop) , it follows from the equations above that $\sigma.\tau = \sum a_ib'_i - \sum a'_ib_i = \iint_X \omega \wedge \eta$. If the forms ω and η are both holomorphic, then their wedge product is zero. This expression shows in that case their coefficient vectors in terms of the symplectic homology basis are orthogonal, i.e. then $\sum a_ib'_i - \sum a'_ib_i = 0$. To find the

coefficients of a loop in terms of the given basis we integrate against the dual form, so $\sigma = \sum a_i A_i + \sum b_j B_j$, where σ is dual to ω if and only if $a_i = \int_{B_i}\omega$ and $b_j = -\int_{A_j}\omega$. To see this, note that $\sigma = \sum a_i A_i + \sum b_j B_j$ yields $\int_{B_k}\omega = \sigma.B_k = \sum a_i A_i.B_k + \sum b_j B_j.B_k = a_k$. The other one is similar.

In this notation, this says if ω and η are both holomorphic then
$\sum \int_{B_i}\omega \int_{A_i}\eta - \sum \int_{A_i}\omega \int_{B_i}\eta = 0$. This is sometimes called Riemann's 1st bilinear relation for one - forms of first kind.

Observe next if ω is holomorphic, then $\bar{\omega} \wedge \omega$ has positive imaginary part, as does its integral, unless $\omega = 0$. A calculation similar to that above shows $\sum \int_{B_i}\omega \int_{A_i}\bar{\omega} > 0$, unless ω is zero. This is sometimes called Riemann's second bilinear relation for forms of first kind. This implies that forms of first kind can be normalized over the A periods as follows.

Corollary: Given a symplectic homology basis A_i, B_j for a Riemann surface X, there is a unique sequence of holomorphic 1 forms $\omega_1,\ldots,\omega_g$, such that $\int_{A_i}\omega_j$ is zero unless i = j, and then it is 1.

proof: The second bilinear relation implies that the A periods of a holomorphic differential are all zero if and only if the differential is zero. Hence the map from the g dimensional space of holomorphic differentials into the dual of the g dimensional complex subspace of $H_1(X,C)$ spanned by the A-cycles, is injective, hence isomorphic. **qed.**

The bilinear relations for forms of 1st and 2nd kind

Now we look at relations for forms of first kind paired with forms of second kind. We keep fixed a particular symplectic homology basis of simple loops A_i, B_j as before, and we note that, given any finite set of points of X, we can arrange for these loops and also the collars around them to be disjoint from the given points.

Let ξ_i be a smooth form Poincare dual to A_i, and let η_j be dual to B_j, again in the sense that integration against the form equals intersection with the dual loop, i.e. $\int_{(\ )}\xi = A.(\ )$, and $\int_{(\ )}\eta = B.(\ )$, as operators on $H_1(X,C)$. Note that all forms of (first and) second kind are locally exact, i.e. their integrals are locally path independent, so they define cohomology classes and can be expressed in cohomology in terms of the basic forms ξ_i, η_j.

Of course one cannot integrate a differential of second kind over a path that passes through a pole, but one can deform the path slightly so as not to pass through the pole. The fact that the residue at the pole is zero implies that the integral of the deformed curve is independent of the deformation. I.e. the integral is not defined over all paths, but every homology class of paths contains paths such that the integral is defined, and the integral is the same over all homologous paths for which it is defined.

Thus if ω is any form of second kind, we can express $\omega = \sum_i a_i \xi_i + b_i \eta_i$, with i summed from 1 to g as usual. Integrating both sides against B_j gives $\int_{B_j}\omega = \sum a_i \int_{B_j}\xi_i + b_i\int_{B_j}\eta_i$

$= \sum_i a_i A_i.B_j + b_i B_i.B_j = a_j$. Similarly, $\int_{A_j}\omega = -b_j$. Thus the coefficients used to express ω in terms of the basis dual to the symplectic basis A_i, B_j are the same as those used to express the loop dual to ω in terms of A_i, B_j.

Now let ω be of first kind and ω' be of second kind with poles at the points $p_1,...,p_d$, and write these forms in cohomology as above with $\omega = \sum_i a_i \xi_i + b_i \eta_i$, $\omega' = \sum_i a'_i \xi_i + b'_i \eta_i$. Then since $\omega - \sum_i a_i \xi_i + b_i \eta_i$ is a smooth form in X with no A or B periods, it is globally exact. Also

$\omega' - \sum_i a'_i \xi_i + b'_i \eta_i$ has no A or B periods, but it is not globally exact in X since it has poles at the points $\{p_k\}$. Nonetheless $\omega' - \sum_i a'_i \xi_i + b'_i \eta_i = dg$, is exact in the complement of the poles, and since the basic differentials are zero near the poles we have $dg = \omega'$ near the poles.

If we remove small open disks around the poles we obtain a Riemann surface S with boundary a disjoint union of circles oriented clockwise around the poles $\{p_k\}$. Then g is a smooth function in S. Since ω and ω' are holomorphic in S, we have $\omega \wedge \omega' = 0$ in S and thus the integral $\iint_S \omega \wedge \omega' = 0$. Now this integral does not equal the intersection pairing of the loops dual to the forms ω and ω', because the integral is taken only over S, and the form ω' is not holomorphic in all of X. But we can relate the integral over the surface with boundary to the intersection number of those cycles as follows.

Recall that $\omega' = \sum_i a'_i \xi_i + b'_i \eta_i + dg$, in S, so we can decompose the previous integral as $0 = \iint_S \omega \wedge \omega' = \iint_S \omega \wedge (\sum_i a'_i \xi_i + b'_i \eta_i) + \iint_S \omega \wedge dg$. Now the forms ξ_i and η_i are zero near the poles so we can extend the first integral on the right over X, i.e. $\iint_S \omega \wedge (\sum_i a'_i \xi_i + b'_i \eta_i) =$ $\iint_X \omega \wedge (\sum_i a'_i \xi_i + b'_i \eta_i)$. Then the earlier calculations apply to give us $\iint_X \omega \wedge (\sum_i a'_i \xi_i + b'_i \eta_i)$ $=$

$\sum_i a_i b'_i - a'_i b_i$, as before.

Thus we have $0 = \iint_S \omega \wedge \omega' = \sum_i a_i b'_i - a'_i b_i + \iint_S \omega \wedge dg$. But we also have $\iint_S \omega \wedge dg =$ $- \iint_S dg \wedge \omega = - \iint_S d(g\omega) = -\int_{\partial S} g\omega$, by Stokes' theorem.

Then since g and thus $g\omega$ is meromorphic on a punctured neighborhood of the poles, the residue theorem gives for this last quantity, $-\int_{\partial S} g\omega = (2\pi i)\sum_k \mathrm{res}_k(g\omega)$, summed over the poles $\{p_k\}$ of ω'. (The minus sign goes away since the orientation of S is opposite to that of the disks surrounding the poles.)

This gives $0 = \iint_S \omega \wedge \omega' = \sum_i a_i b'_i - a'_i b_i + (2\pi i)\sum_k \mathrm{res}_k(g\omega)$, so

$\sum_i a_i b'_i - a'_i b_i = - (2\pi i)\sum_k \mathrm{res}_k(g\omega)$. Now if $\omega' = (c_k z^{-2} + \text{holo})dz$ near the pole p_k, then $g = -c_k z^{-1}$ near p_k, and so $g\omega = (-c_k e_k z^{-1} + \text{holo})dz$ near p_k, where e_k = the value of ω at p_k in terms of the local coordinate z. I.e. $e_k = (\omega/dz)(p_k)$, or by abusing notation simply $e_k = \omega(p_k)$. Hence assuming $\omega' = (c_k z^{-2} + \text{holo})dz$ near the pole p_k, we have

(*) $\sum_i (a_i b'_i - a'_i b_i) = (2\pi i)\sum_k c_k \omega(p_k)$.

This formula appears explicitly in [GH], p. 241, (give or take a minus sign), and is formula (1) in the original paper of Roch, [Crelle, 1865]. [Unfortunately the translation in: A source book of classical analysis, p.202, Harvard Univ. Press, ed. Garrett Birkhoff, not only contains misprints which render the formula incorrect, but also omits arguments which might enable the reader to catch the errors. Hence the original version is preferable even for readers with minimal German.]

Now apply (*) to the case of a standard differential ω' of second kind at p_k with all A periods zero, and let $\omega = \omega_j$ = the standard differential of first kind with all A periods zero except over A_j where the period is 1. Then we have all $b_i = 0$ except $b_j = 1$, and all $b'_i = 0$. This gives $-\int_{B_j}\omega' = -a'_j = (2\pi i)\sum_k c_k \omega_j(p_k)$, where $\omega_j(p_k)$ is the value of ω_j at p_k in terms of the local coordinate z at p_k.

Now we re - examine the integration map W(D)-->C^g in Riemann's proof above of his inequality, by computing its matrix in the standard basis for W(D) given by $\omega'_k = (z^{-2}$ +holo.)dz near the pole p_k, and holo. elsewhere. Then the map takes ω'_k to $\int_{B_j}\omega'_k = -2\pi i\, \omega_j(p_k)$ = the (j,k) entry of the matrix. We will compute the rank of this matrix M two ways.

A relation among the rows would be a coordinate vector $(t_1,\ldots,t_g)$ orthogonal to all the columns, i.e. such that for all k, we have $\sum_j t_j\, \omega_j(p_k) = 0$. This is precisely a holomorphic 1 form $\sum_j t_j\, \omega_j$ that vanishes at all the points p_k. Thus if we denote the space of such forms by L(K-D) where D is the divisor $D = \sum_k p_k$, then rank(M) = g - dimL(K-D). It follows that dimker(M) = dim(domain(M)) - rank(M) = d - g + dimL(K-D).

If we recall that this kernel is precisely the space of differentials df for all f in L(D), we have dimL(D) -1 = d - g + dimL(K-D). Rewriting gives

RRT: dimL(D) = d+1 - g + dimL(K-D).

It is enlightening also to compute the number of relations among the columns, i.e. the vectors orthogonal to the rows. If (c_k) is such a vector, then we have for all j = 1,...,g, that $\sum_k c_k\, \omega_j(p_k) = 0$. We interpret this as follows: if the vector (c_k) corresponds to the unique differential ω' of second kind having form $(c_k z^{-2} + \text{holo})dz$ near p_k for every k, and having zero A periods, then ω' is exact if and only if for all j = 1,...,g, the B_j periods $\int_{B_j}\omega'$ are also zero. But we have computed $\int_{B_j}\omega' = (-2\pi i)\sum_k c_k\, \omega_j(p_k)$. Thus a relation $\sum_k c_k\, \omega_j(p_k) = 0$ among the columns, is equivalent to a differential in W(D) which is exact, i.e. to a differential df, for f in L(D). Thus we see that the row rank, is d - (dim(L(D)-1). Since this equals the column rank we have g - dimL(K-D) = d+1 - dimL(D), so again dimL(D) = d+1-g+dimL(K-D).

Summary of the classical proof by Riemann and Roch

Let $D = p_1 + ...+p_d$ be a divisor of distinct points on a compact connected Riemann surface X of genus g, and let L(D) be the space of meromorphic functions on X with at worst simple poles contained in the set $\{p_1,...,p_d\}$. For each point p_j let μ_j be a meromorphic differential with exactly one pole, a double pole at p_j, and hence residue zero at p_j. Let $w_1,...,w_g$ be basis for the holomorphic differentials on X. Then for every function f in L(D), the differential df belongs to the linear space V(D) of differentials with basis $\{\mu_1,...,\mu_d,w_1,...w_g\}$. Indeed the subspace of differentials of form df in V(D) consist exactly of those differentials in V(D) with zero period, i.e. zero integral, around all loops $\{A_1,...,A_g,B_1,...,B_g\}$ in a homology basis for X. Thus the period matrix defines a linear map
V(D)--->C^{2g} whose kernel is isomorphic to L(D)/C. Since dimV(D) = d+g, the fundamental theorem of linear algebra implies
$d+g -(2g) = d-g \leq \dim L(D)/C \leq d+g$.

Since the period map is injective on holomorphic differentials, in fact the kernel L(D)/C does not meet the g dimensional subspace of V(D) spanned by $w_1,...,w_g$. Hence we get a better upper bound, $\dim L(D)/C \leq d$,
i.e. $d-g \leq \dim L(D)-1 \leq d$. **This is Riemann's part of the theorem**.

Roch then analyzed the period matrix defining the map V(D)--->C^{2g} to compute its cokernel. First of all he normalized the meromorphic differentials $\mu_1,...,\mu_d$ to have all A-periods equal to zero by subtracting suitable linear combinations of the w_j, and defined W(D) to be the d dimensional space of meromorphic differentials with basis $\{\mu_1,...\mu_d\}$. Then differentiation maps L(D) into W(D) and the image, isomorphic to L(D)/C, equals those differentials in W(D) whose B periods are also zero. Thus the B-period map S(D):W(D)--->C^g, again has kernel isomorphic to L(D)/C. Since dimW(D) = d, again we get $d-g \leq \dim L(D)-1 \leq d$.

Next Roch computes explicitly the rank of the B-period matrix S(D). For this he normalized also each holomorphic differentials w_j to have all A-periods zero except over A_j where the period is 1. Then he observed that by residue calculus Riemann's bilinear relations as above show that the integral of μ_k over B_j equals $-2\pi i\, w_j(p_k)$. Hence Riemann's matrix S(D) = $[\ \int_{B_j} \mu_k\ ]$ of periods of meromorphic differentials, is proportional to Roch's matrix T(D) = $[w_j(p_k)]$ of values of holomorphic differentials, (which has come to be called the "Brill Noether" matrix, apparently just because they displayed it in a larger format). It is then elementary that the rank of Roch's matrix equals g - dim(K(-D)), where K(-D) is the space of holomorphic differentials vanishing at every point $p_1,...,p_d$ of D. I.e. the kernel L(D)/C has dimension d - (g-dim(K(-D))) = d-g + dim(K(-D)), so dim(L(D)/C) = d-g + dim(K(-D)), i.e. dimL(D) = d+1-g + dim(K(-D)). **This is the full classical Riemann Roch theorem.**

After discussing sheaf cohomology below, we will give the sheaf theoretic translations of the matrices of Riemann and Roch.

Riemann's derivation of the "Brill Noether" number

Riemann's argument already shows that, in the symmetric product $X_d = X^d/Sym(d)$ parametrizing effective (i.e. positive) divisors of degree d, the subvariety X(r,d) of divisors D of degree d and dimL(D) > r on a given curve X, has a local determinantal structure. I.e. his calculation says that dimL(D) - 1 = dim ker[S(D)], where S(D) is a (2g) by (g+d) "period matrix" for differentials of second kind, parametrized by the divisor D. Thus X(r,d) is the locus of divisors D such that rank(S(D)) ≤ (d-r+g).

When r = 1, Riemann himself explicitly says this, and immediately concludes that a generic curve X of genus g, is expected to have a non constant meromorphic function with at most d poles only if d ≥ (g/2) + 1, which he deduces from the inequality (d-1) ≥ (g+1-d) (= codimension of the rank (d-1+g) locus, in the space of (2g) by (g+d) matrices). The similar estimate (d-r) ≥ r(g+r-d) gives the "Brill - Noether" estimate for X(r,d) to be non empty for all curves of genus g.

The point here is that Riemann's matrix S(D) defines a map from the divisor variety X_d into matrix space Mat(2g, g+d), and we want a condition implying that the image meets the locus where rankS(D) ≤ d-r+g. Since this rank locus has codimension r(g+r-d), X_d has dimension d, and the intersection will have dimension r when it is non empty, we expect this to occur when (d-r) ≥ r(g+r-d). Equivalently we have g ≥ (r+1)(g-d+r), the so called "Brill - Noether" criterion predicting the existence of a divisor D of degree d with dimL(D) > r, on all curves X of genus g.

Interpreting Roch's map as a residue pairing

It is interesting to notice the following interpretation of the row rank of Roch's matrix. Given any vector (c_k) in C^d, we may consider it as corresponding to a "principal part" P for the divisor $D = \sum_k p_k$, where near each point p_k we take the principal part $P_k = c_k z^{-1}$ in the local coordinate z near p_k. Then we see how to state the criterion for existence of a meromorphic function having the principal parts P purely in terms of P and the holomorphic differentials. This rewrites Riemann's exact sequence as follows:

$$0 \to C \to L(D) \to Prin_D \to K^*,$$

where the map $L(D) \to Prin_D$ takes a meromorphic function to its principal part, supported on D. The map $Prin_D \to K^*$, which replaces Riemann's period map $W(D) \to C^g$, takes a family of principal parts $\{P_i\}$ supported in D, to the linear functional on holomorphic differentials sending

ω to $\Sigma_i \, res_i(P_i \omega)$.

This interprets the map $W(D) \to C^g$ as a residue pairing $P_D \to K^*$.

The residue condition for existence of a meromorphic function

We have seen that a meromorphic function exists with principal parts P if and only if the unique standard differential ω' of second kind associated to P is exact, if and only if for all j the

B_j period $\int_{B_j} \omega'$ = 0, if and only if for all j, $\sum_k c_k \omega_j(p_k) = 0$, if and only if for all j the sum of the residues at the points $\{p_k\}$ of the family of local differential forms $P_k \omega_j$ is zero.

This just says $\sum_k \text{res}(P_k \omega_j) = 0$ for all j, and since the ω_j are a basis for the space of global holomorphic differentials, this holds if and only for every holomorphic differential ω we have $\sum_k \text{res}(P_k \omega) = 0$.

Thus the vanishing condition on the sum of the residues both characterizes existence of a meromorphic differential, and is necessary and sufficient for the existence of a meromorphic function. Given a finite collection $\{P_k\}$ of principal parts at a finite set of points $\{p_k\}$ of X, there is a meromorphic function having precisely those poles and those principal parts if and only if for every holomorphic differential ω on X, we have
$\sum_k \text{res}(P_k \omega) = 0$.

With a little more reasoning, mostly formal, one can remove the restriction to effective divisors. I.e. if D is any divisor at all, and L(D) is the space of meromorphic functions such that div(f) + D ≥ 0, (or f = 0), and L(K-D) is the space of meromorphic differentials ω such that div(ω)-D ≥ 0, or equivalently the space of meromorphic functions f such that div(f)+K-D ≥ 0, where now K denotes the divisor of some fixed differential, then again we have dimL(D) = 1+deg(D) -g + dimL(K-D).

When D is not effective, we no longer have Riemann's inequalities, since then deg(D)≤ g-1, so the lower bound dimL(D) ≥ 0 ≥ 1+deg(D) -g is trivial, and the upper bounds dimL(K-D) ≤ g and dimL(D) ≤ 1+deg(D) can be false, e.g. if deg(D) < 0.

The residue form of the Riemann Roch exact sequence

Let k(X) denote the infinite dimensional complex vector space of all rational functions on X, and let Prin(X) denote the infinite dimensional space of all finite vectors of principal parts on X. I.e. Prin(X) is the direct sum over all points p of X of the quotient spaces k(X)/O_p(X), where O_p(X) is the subspace of k(X) consisting of rational functions which are regular at p. Define the map Prin(X)-->H^0(K)*, where H0(K) is the space of all holomorphic differentials on X, and the star denotes dual space, which takes a vector of principal parts $\{P_k\}$ to the functional whose value on ω is the sum of the residues $\sum_k \text{res}(P_k \omega)$. Let k(X)-->Prin(X) be the natural map taking a rational function to its vector of principal parts. Then we have just shown that the Riemann Roch theorem (and the residue theorem) implies this sequence is exact:

0-->C-->k(X)-->Prin(X)-->H^0(K)*-->0,

where the last map is given by the residue pairing Prin(X) x H^0(K)-->C.

Thus there are exactly g linear conditions that must be satisfied before a given vector of principal parts will arise from a global meromorphic function. Roch's part of the theorem determines to what extent those conditions are independent for a given divisor. An effective divisor D defines a finite dimensional subspace Prin(D) of Prin(X) consisting of principal parts whose divisors are bounded below by -D, and then the induced sequence is exact: 0-->C-->L(D)-->Prin(D)-->H^0(K)*.

There is no zero on the right since the equations in $H^0(K)$ may not be independent on the finite dimensional subspace Prin(D). By Roch, a holomorphic differential ω defines a trivial equation on Prin(D) if and only if ω vanishes at all points of D (assuming D consists of distinct points), so we see more precisely the following sequence is exact:

Riemann Roch theorem:
0-->C-->L(D)-->Prin(D)-->$(H^0(K\text{-}D))^\perp$-->0, is exact,
where $(H^0(K\text{-}D))$ is the subspace of $H^0(K)$ consisting of holomorphic differentials which vanish on D, and $(H^0(K\text{-}D))^\perp$ is the subspace of $H^0(K)^*$ orthogonal to that subspace.

This is the full RRT for effective D. Thus dimL(D) + dim$(H^0(K\text{-}D))^\perp$ = dim(C) + dimPrin(D), which gives dimL(D) + g - dimL(K-D) = 1 + deg(D),
i.e. dimL(D) = 1+deg(D) -g + dimL(K-D), as before.

I.e. Riemann's original exact sequence:
0-->C-->L(D)-->W(D)-->C^g, where the last map is the B -period map,
becomes by the reciprocity relations the sequence:
0-->C-->L(D)-->Prin(D)-->$H^0(K)^*$.

Then the cokernel of this sequence is computed by:
0-->C-->L(D)-->Prin(D)-->$H^0(K)^*$-->$[H^0(K\text{-}D)]^*$-->0.
The exactness of this sequence is one statement of the classical RRT.

Remark: The sequence 0-->C-->k(X)-->Prin(X)-->$H^0(K)^*$-->0 is just the exact cohomology sequence 0-->C-->k(X)-->Prin(X)-->$H^1(O)$-->0 of the sheaf sequence 0-->O-->k(X)-->Prin(X)--->0 which defines the cohomology of O (cohomology is reviewed below) plus the residue isomorphism
$H^1(O) = H^0(K)^*$.

Chapter IV. The general RR problem, the modern approach

part (1): computing the "index" chi(D). (Hirzebruch RR)

As observed above, one can view the problem of computing dim(L(D)) as falling into separate parts, first (HRR) find an integer chi(D) closely related to dim(L(D)) but which is a deformation invariant, and find a topological formula for this invariant. Historically I believe Serre proved chi(D) is a topological invariant before Hirzebruch found the explicit formula for chi(D) in terms of chern cohomology classes. Then (by Serre or Kodaira vanishing) find conditions under which chi(D) = dim(L(D)), i.e. when the extra terms involved in chi(D) are all zero.

The relative difficulty of the steps depends on the definition of chi. In the case of curves, if we define chi(D) = dimL(D) - dim(K(-D)) as Hirzebruch does, then topological invariance is difficult, but since deg(K) = 2g-2 by Hopf's theorem equating the number of zeroes of a general smooth vector field with the euler characteristic, a vanishing condition on K(-D) is easy, since a holomorphic differential can never have more than 2g-2 zeroes, so chi(D) = dimL(D) when d > 2g-2.

On the other hand, if we define chi(D) in terms of sheaf cohomology (discussed briefly below) as $h^0(D) - h^1(D)$, then the proof that the difference chi(D) - chi(O) = d, hence is a topological invariant, is easy. I.e. sheaf theory implies immediately that the difference chi(D) - chi(O) increases by 1 for each point added to D. Since the formula is clearly true when D = 0, we are done by induction. Then the proof that just chi(O) is a topological invariant equal to 1-g, can also be done by induction. We will give this argument below.

In the cohomological approach, the deep content of RRT for curves is in the proof of "Serre duality", which implies $h^1(D)$ = dim(K(-D)), and gives both the topological meaning of g, and a criterion for chi(D) to equal dimL(D), i.e. for the vanishing of $h^1(D)$. The more limited result (HRR) that chi(O) = 1-g, given below, is inspired by the axiomatic proofs of the general HRR theorem of Washnitzer and Fulton.

The usefulness of RRT is that it tells you that for a given divisor D, you should usually be able to compute chi(D) fairly easily by HRR, and in some cases, i.e. when D is "large", that number will be equal to dimL(D) by Kodaira vanishing. In the cases of curves and surfaces, "Serre duality" which interprets the top cohomology group of D as dual to L(K-D) gives more precise information on dimL(D) for arbitrary D.

We will not say much more about proofs of "vanishing" results, but will discuss some of the modern proofs for computing chi(D), i.e. HRR.
That is, we prove a few things using sheaf cohomology, instead of integrals or Riemann's results on existence of differentials. We concentrate on results that follow easily from formal properties.

HRR for curves, including a topological computation of chi(O).

We will use the theory of cohomology for sheaves on curves, e.g. for line bundles associated to divisors which we recall next.

Review of cohomology of a line bundle on a curve.

Briefly, if M is any line bundle on X, then there is an exact sequence like the one above: 0-->H^0(M)-->Rat(M)-->Prin(M), where Rat(M) is the space of rational sections of M, and Prin(M) is the direct sum over all points p in X of the quotients Rat(M)/{rational sections regular at p}. If M = O(D) where D is a divisor, then H^0(M) is isomorphic to L(D). The cokernel of the map Rat(M)-->Prin(M) is finite dimensional and denoted H^1(M) or H^1(D), so 0-->H^0(M)-->Rat(M)-->Prin(M)-->H^1(M)-->0 is exact.

If D≥0, this agrees with the definition above where H^1(O) and H^1(D) are the groups making the sequences 0-->C->k(X)-->Prin(X)-->H^1(O)-->0 and 0-->C->L(D)-->Prin(D)-->H^1(O)-->H^1(D)-->0 exact. Cohomology spaces can also be defined for other sheaves as well, e.g. for the quotient of a map of line bundles.

For example if M = O = the sheaf of regular functions corresponding to the trivial divisor 0, then we have the exact sequence
0-->H^0(O)-->Rat(O)-->Prin(O)-->H^1(O)-->0. But the only globally regular functions are constants, and Rat(O) = k(X), and Prin(O) = Prin(X), so we have 0-->C-->k(X)-->Prin(X)-->H^1(O)-->0. If we assume RRT, i.e. the exactness of the sequence 0-->C-->k(X)-->Prin(X)-->H^0(K)*-->0, we could deduce that H^1(O)is isomorphic to H^0(K)* which by RRT has dimension g. However we will give a topological computation that dimH^1(O) = g below, without assuming RRT.

The long exact sequence property for sheaf cohomology

For any short exact sequence 0-->M-->N-->R-->0, of sheaves on a curve X, there is a long exact sequence of cohomology groups:
0-->H^0(M)-->H^0(N)-->H^0(R)-->H^1(M)-->H^1(N)-->H^1(R)-->0. It follows from the rank - nullity theorem that the alternating sum of the dimensions of these spaces is zero, i.e. that chi(M) + chi(R) = chi(N).

The most fundamental short exact sequence of sheaves on a curve arises from an effective divisor D by letting O(-D) = I_D be the ideal sheaf of regular functions vanishing at points of D. Then we have the exact sequence 0-->O(-D)-->O-->O$|_D$-->0, where O$|_D$ is the sheaf of regular functions defined only on the points of D, hence its space of global sections is isomorphic to C^d where d = deg(D). The two non trivial maps in the sequence are inclusion and restriction.

(weak RR) Proposition: chi(D) - chi(O) = deg(D), for any effective divisor D on a curve.
proof: This follows from the basic sequence and the additivity of chi. We have 0-->I_D-->O_C-->O_D-->0, where I_D is the ideal sheaf of the subscheme D in C. Then we know that I_D is isomorphic to O_C(-D) so we have exact sequences: 0-->O_C(-D)-->O_C-->O_D-->0, which becomes
0-->O_C-->O_C(D)-->$O_D|_D$-->0 after tensoring by O_C(D).

There is no twisting for a line bundle on a finite set, i.e. no obstruction to the existence of sections, so the global sections of $O_D|_D$ have dimension equal to deg(D). There are also no higher cohomology groups for such a zero dimensional scheme so chi($O_D|_D$) = deg(D). Thus by additivity of chi we have chi(O_C(D)) - chi(O_C) = deg(D). **qed.**

Note this is an inductive argument, on dimension. I.e. the computation of chi(D)-chi(O) reduces to a computation on a finite set, which is trivial. In a similar way, the analogous computation on a surface reduces to the case of RRT for a curve, modulo the adjunction formula relating differentials on the surface to those on the curve.

After the easy proposition above, HRR on a curve consists in completing the computation of chi(D) by computing chi(O) as 1-g, where g is the topological genus of the curve. One way to do that is inductive, i.e. show that both formulas chi(O) and (1-g) agree for curves of

lower genus, and then show for a curve of higher genus that:
(i) both formulas stay constant under degeneration to a union of curves of lower genus, and
(ii) both formulas are additive in the same way over lower genus components.

We will state this more precisely in properties A, B below.

Proposition: If we define chi(O) = h^0(O) - h^1(O), then for any smooth complex plane curve X of degree d, we have chi(O) = 1 - (d-1)(d-2)/2 = 1-g, where g is the topological genus of X. Thus h^1(O) = g.

Remark: We will prove this formula for smooth plane curves, and then use that case as a tool to prove it for all smooth curves, planar or not, by projecting them into the plane, and then taking into account any nodes acquired by the projection and their effect on both sides of the formula.

proof of proposition. To show that chi(O) = 1 - (d-1)(d-2)/2 = 1-g, we will compute how chi(O) varies as a curve moves in a linear series.

lemma A: If X,Y are two curves on a smooth surface S, and if X,Y are linearly equivalent as divisors on S, then chi(O_X) = chi(O_Y).
Remark: This says in some sense chi(O) is a deformation invariant, at least for linear deformations.
proof: Since the line bundles O_S(-X) and O_S(-Y) are isomorphic on S, the invariants chi(O_S(-X)) and chi(O_S(-Y)) are equal. By the usual exact sheaf sequence 0--> O_S(-X)--->O_S--->O_X--->0, and the analogous one for Y, plus the additivity of chi, we get that chi(O_X) = chi(O_S) - chi(O_S(-X) = chi(O_S) - chi(O_S(-Y)) = chi(O_Y). **qed.**

lemma B: Now suppose that Y, Y' are curves on a smooth surface S, and that Y and Y' meet transversely at precisely n points. Then we claim chi($O_{Y+Y'}$) = chi(O_Y) + chi($O_{Y'}$) - n.
proof: Consider the sequence 0-->$O_{Y+Y'}$-->O_Y + $O_{Y'}$-->$O_{Y.Y'}$-->0, induced by the map from the disjoint union of Y,Y', to their union Y+Y' on S, and where the map to $O_{Y.Y'}$ is the difference of the two restrictions, from Y and from Y', to the intersection of Y and Y'. The additivity of chi then implies the desired relation, i.e. chi(O_Y) + chi($O_{Y'}$) = chi(O_Y + $O_{Y'}$) = chi($O_{Y+Y'}$) + chi($O_{Y.Y'}$) = chi($O_{Y+Y'}$) + n. Thus chi($O_{Y+Y'}$) = chi(O_Y) + chi($O_{Y'}$) - n. **qed.**

Now that we know how the function chi(O) behaves under linear degeneration, all we need is to find a formula that behaves this way, and it must be the formula for chi(O). Since we secretly know also that chi(O) = 1-g, it will suffice to have a formula for the genus of a plane curve.

Lemma: A smooth plane curve X of degree d has genus (d-1)(d-2)/2.
proof: Note that if d = 1 or 2, then X is isomorphic to P^1 which is a sphere of genus zero. Now let a smooth curve of degree d ≥ 3 degenerate into a smooth curve Y of degree d-1 plus a transverse line L meeting Y in d-1 distinct points. Then Y has genus (d-2)(d-3)/2 by induction, and L is a sphere which meets Y in exactly d-1 points transversely. A nbhd of each intersection point on Y+L looks topologically like the union of two discs, i.e. like a real quadratic cone x^2+y^2

$= z^2$.

We assume the plausible fact that as Y+L moves in a linear system back to become the smooth curve X of degree d, that topologically this replaces the cone $x^2+y^2 = z^2$ by the hyperboloid $x^2+y^2 = z^2 + a$, for a > 0. (I.e. acquiring a node occurs by letting a-->0 in $x^2+y^2 = z^2 + a$.) This surgery adds d-2 handles to Y, so that g(X) = g(Y) + d-2. Thus g(X) = (d-2)(d-3)/2 + (d-2) = $(d^2-5d+6)/2 + (2d-4)/2 = (d^2-3d+2)/2$ = (d-1)(d-2)/2, as claimed. **qed.**

This motivates the statement of the next result.
Corollary: If X is a smooth plane curve of degree d, then $chi(O_X) = 1 - (d-1)(d-2)/2$.
proof: Induction on d. If d = 2, then the smooth conic X moves in a linear series also containing a union Y of two lines $Y_1 + Y_2$, where each line is isomorphic to X. Then by lemmas A,B above, we have $chi(X) = chi(Y_1)+chi(Y_2) - 1$ = chi(X)+chi(X)-1, hence chi(X) = 1. This proves the case d = 2, and since a smooth curve of degree d = 1 is isomorphic to one of degree 2, we also obtain the formula for degree d=1.

Now assume d ≥ 3 and that we have proved the formula for smooth curves of degree < d. A smooth degree d curve X moves in a linear series that also contains a curve of form $Y = Y_1+Y_2$, where Y_1 is smooth of degree d-1, and Y_2 is a line meeting Y_1 transversely in d-1 distinct points. Then lemmas A, B and induction give us that $chi(O_X) = chi(O_Y) = chi(O_{Y1})+chi(O_{Y2})-(d-1)$ = 1-(d-2)(d-3)/2 + 1 - (d-1) = 1-(d-1)(d-2)/2, as desired. **qed.**

Corollary: If Y is an irreducible plane curve of degree d, with n nodes and no other singularities, and X is the normalization of Y, then
$chi(O_X) = n+1 - (d-1)(d-2)/2$.
proof: Blow up the plane at the nodes of Y, obtaining a smooth surface S, on which the total transform of Y is the union of the smooth curves X and X', meeting transversely at 2n points, where X' is a disjoint union of n lines. Let E be a smooth plane curve of degree d which does not pass through the n points, so that E also lies on S and is linearly equivalent to X+X' there. Then by the formulas above, applied to the blown up plane S, (and which do not use connectivity of X'), we have 1 - (d-1)(d-2)/2 = chi(E) = chi(X+X') = chi(X) + chi(X') - 2n = chi(X) + n -2n = chi(X) - n. So chi(X) = n+1 - (d-1)(d-2)/2, as claimed. **qed.**

Cor: For a smooth compact connected curve X of genus g, $chi(O_X) = 1-g$.
proof: It is known that X is isomorphic to the normalization of an irreducible plane curve of some degree d with n nodes where g(X) = (d-1)(d-2)/2 - n. I.e. a smooth curve of degree d has genus (d-1)(d-2)/2 and each node lowers the genus by one. Thus for such a curve X we have $chi(O_X) = n+1 -(d-1)(d-2)/2 = 1 - g(C)$. **qed.**

Sheaf theoretic versions of the original Riemann and Roch maps

I claim the argument above essentially recaptures just Riemann's part of the original RR theorem. Recall Riemann's (g by d) "B-period" map
$S(D):W(D)\text{--->}C^g$ with kernel isomorphic to L(D)/C, where the only omission was he did not calculate the cokernel. The sheaf version of this period matrix is essentially the coboundary map $H^0(X,O(D)|_D)\text{-->}H^1(X,O)$, induced from the sheaf sequence $0\text{-->}O\text{-->}O(D)\text{-->}O(D)|_D\text{-->}0$, used above to compute chi(D) - chi(O), while Roch's evaluation map is the transpose of the restriction map $H^0(X,O(K))\text{-->}H^0(X,O(K)|_D)$, induced from the sequence
$0\text{-->}O(K\text{-}D)\text{-->}O(K)\text{-->}O(K)|_D\text{-->}0$. This is fairly clear for Roch's map.

For the period map (in Roch's normalization), the source is the space of meromorphic differentials with poles only at the points of D, of order at most 2 and with all residues zero, modulo holomorphic differentials. By the converse of the residue theorem, this is equivalent to the space of possible principal parts of such differentials at the points of D, i.e. the sections of a skyscaper sheaf supported on D. Now the sections $H^0(X,O(D)|_D)$ of the skyscraper sheaf $O(D)|_D$, is the space of possible principal parts for meromorphic functions with pole divisor supported in D, and differentiation takes this space by the converse of the residue theorem to one isomorphic to the source space for the period map S(D).

The target for Riemann's map is the orthogonal complement $(H_A)^{perp}$ in $H^1(X,C)$ of the span H_A of the of A - cycles in $H_1(X,C)$. Since the subspaces $(H_A)^{perp}$ and $H^0(K)$ are complementary in $H^1(X,C)$, we may regard $(H_A)^{perp}$ as naturally isomorphic to the quotient $H^1(X,C)/H^0(K) = H^1(X,O)$. I.e. the period map on meromorphic differentials goes into $H^1(X,C)$, so the period map on meromorphic differentials modulo holomorphic ones (i.e. normalized meromorphic differentials), goes into $H^1(X,C)/H^0(K) = H^1(X,O)$.

So the coboundary map $H^0(X,O(D)|_D)\text{-->}H^1(X,O)$ may be thought of as the composition taking a principal part in $H^0(X,O(D)|_D)$ by differentiation to a principal part for a meromorphic differential of second kind, then to a unique such differential (modulo holomorphic ones), then to a cohomology class in $H^1(X,C)/H^0(K) = H^1(X,O)$.

As we have seen, Riemann knew the kernel of this map was isomorphic to L(D)/C, and that the target space has dimension g, but did not compute the cokernel $H^1(D)$. To be honest we should admit that calling the cokernel $H^1(D)$ is not computing it but merely giving it a name, so with this sequence we are in the same situation as Riemann.

However what Roch essentially showed is that the exact cohomology sequences coming from the two sheaf sequences
$0\text{-->}O\text{-->}O(D)\text{-->}O(D)|_D\text{-->}0$, and $0\text{-->}O(K\text{-}D)\text{-->}O(K)\text{-->}O(K)|_D\text{-->}0$, are "dual" to each other, i.e. the sequence of maps:

(1)

0-->H^0(O)-->H^0(O(D))-->H^0(O(D)$|_D$)-->H^1(O)-->H^1(O(D))-->0,

is dual to the following sequence:
(2) 0-->H^0(O(K-D))-->H^0(O(K))-->H^0(O(K)$|_D$)-->H^1(O(K-D))-->H^1(O(K))-->0.

I.e. Roch showed the cokernel H^1(D) of the coboundary H^0(O(D)$|_D$)-->H^1(O), is dual to the kernel H^0(O(K-D)) of the evaluation H^0(O(K))-->H^0(O(K)$|_D$). Since this duality result is all we need to enhance the argument above for the equation chi(D) = d +1-g, to obtain the full RR equation, we sketch next an algebraic sheaf version of that duality, taken from Serre, and using an idea of Weil.

Serre's and Weil's algebraic approach to duality via residues

The fact that the sequence 0-->C-->k(X)-->Prin(X)-->H^0(K)*-->0 is exact, via the residue pairing is the appropriate point of view in trying to prove the RRT algebraically. In one approach, Serre recovers the theory of residues purely algebraically, by pulling back direct calculations from P^1 to branched covers of P^1, and verifies again that they give necessary and sufficient conditions for a vector of principal parts to come from a meromorphic function. This point of view seems to have been introduced by Andre Weil who used a universal space parametrizing principal parts called “adeles”, or “repartitions” by Serre.

I will briefly review the proof as presented by Serre in Groupes algebriques et corps de classes, (and I recommend reading Serre). Let k(X) be the field of rational functions on the curve X, and let Prin(X) be the space of principal parts, i.e. Prin(X) is the direct sum over all points p of X of the quotients k(K)/O_p(X), where O_p(X) is the local ring of rational functions regular at p. An element of k(K)/O_p(X) should be thought of as a principal part at p for a rational function. Then there is an exact sequence 0-->C->k(X)-->Prin(X), where C is the complex numbers and k(X)-->Prin(X) takes a function f to its principal part at each point. Then the cokernel of this map is called H^1(O), the obstruction group measuring which principal parts occur from global rational functions. So we have an exact sequence: 0-->C->k(X)-->Prin(X)-->H^1(O)-->0. A basic result says H^1(O) is finite dimensional over the algebraically closed base field k.

If D is any effective divisor, define L(D) the finite dimensional subspace of k(X) of rational functions with pole divisors bounded below by -D, and Prin(D) to be the finite dimensional subspace of Prin(X) of principal parts bounded below by -D. Then the sequence above restricts to 0-->C->L(D)-->Prin(D)-->H^1(O). Then H^1(D) is by definition the cokernel of the last map, i.e. the quotient H^1(O)/ImPrin(D).

Now define the space R of repartitions, where an element of R is an assignment to each point of an element of k(X), and where all but a finite number of points p are assigned elements in O_p(X). Note that the space Prin(X) is a subspace of the product of all quotients k(K)/O_p(X).

The product of k(X) over all points maps to the product of all quotients $k(K)/O_p(X)$, and R is the inverse image of Prin(X) under this map. There are canonical maps k(X)-->R-->Prin(X).

For any divisor D, let R(D) be the subspace of R such that at every point the order of the function there is bounded below by the order of -D at that point. Then we have R/[R(D)+k(X)] = $H^1(D)$. For example note that the natural map R-->Prin(X) has kernel R(0), that Prin(X) = R/R(0), and hence $H^1(O)$ = Prin(X)/k(X) = R/[R(0)+k(X)].

We want to show for all D that $H^1(D)^* = H^0(K-D)$. The advantage of this approach is that all the groups $H^1(D)$ are quotients of the same space R/k(X). Hence their duals are all subspaces of the dual of R/k(X). In fact one shows that the union for all divisors D, not just effective ones, of the dual spaces $H^1(D)^*$ is the dual space of R/k(X). Then one considers Rat(K) the space of all rational differentials on X, isomorphic to k(X).

The key step is the residue pairing R x Rat(K)-->k, taking a repartition P and a differential w to the sum of the residues of Pw. After proving the residue theorem algebraically, one has that the kernel of this pairing on the left contains k(X). Then there is an induced pairing of
R/k(X) x Rat(K) -- >k, and an induced map Rat(K)-->$(R/k(X))^*$.

Then Serre shows that $(R/k(X))^*$ is a vector space over not just k, but over k(X), and in fact, it has dimension ≤ 1 over k(X), the same dimension as Rat(K). Then to show the map Rat(K)-->$(R/k(X))^*$ is an isomorphism it suffices to show it is not zero. Then one checks the isomorphism restricts to isomorphisms of the various subspaces
$H^0(K-D)$-->$H^1(D)^*$.

Proof of Hirzebruch Riemann Roch for smooth surfaces S_d in P^3.

We follow the same approach as for curves. I.e. we define cohomology groups for sheaves on surfaces, in particular for divisors and line bundles. This time for a sheaf M there are groups $H^0(M)$, $H^1(M)$, $H^2(M)$, and we define chi(S,M) = dim$H^0(M)$ - dim$H^1(M)$ + dim$H^2(M)$ = $h^0(M)-h^1(M)+h^0(M)$. In particular chi(S) = chi(S,O) = $h^0(O)-h^1(O)+h^0(O)$.

Then the same exact sequences prove that chi of a union of transverse surfaces in P^3 is additive in the same way as for curves. If a smooth surface of degree d moves in a linear series to become a transverse union of a smooth surface of degree d-1 and a plane, we get chi(S_d) = chi(S_{d-1}) + chi(P^2) - chi(C_{d-1}) where C_{d-1} is a smooth plane curve of degree d-1. A direct computation with Cech cohomology shows chi(P^2) = 1, so this determines chi(S_d) recursively for all d.

Topological euler characteristic of a smooth surface in P^3

Now by taking a Lefschetz pencil of planes through a general fixed line L in P^3, it is known we fiber a smooth surface S of degree d by plane curves which are generically smooth of degree d, but with exactly $d(d-1)^2$ singular curve fibers each having exactly one node. If we blow up the surface S at its d intersection points with L, to obtain a new surface T, we get a morphism $T \dashrightarrow P^1$ with those same plane curves as fibers. Then by "multiplicativity" of e as in the cubic surface example above, the euler characteristic $e(T) = 2(2 - 2g) + d(d-1)^2 = 2(2-(d-1)(d-2)) + d(d-1)^2$, where $g = (d-1)(d-2)/2$ is the genus of a generic fiber of T. Since blowing up a surface adds one homology 2 cycle at each point blown up, we have

$$e(S) = e(T) - d = 2(2 - (d-1)(d-2)) + d(d-1)^2 - d = d(d^2-4d + 6).$$

This direct computation agrees, happily, with the one we derived earlier from Noether's formula and completeness of the adjunction series.

Canonical class of a smooth surface in P^3

Now we are going to assume we know the algebraic computation of regular differentials on an embedded surface, i.e. the adjunction formula, that a canonical class on S_d is cut out by surfaces of degree d-4, hence the self intersection of the canonical class K on S is $K^2 = d(d-4)^2$. Thus we have $(1/12)(e(S_d) + K^2) = d(d^2-6d+11)/6 = f(d)$.

Noether's formula for a smooth surface S_d in P^3

It is easily proved by induction that the formula $d(d^2-6d+11)/6 = f(d)$ satisfies the same recursive relations as did the formula for chi(O). I.e., $f(1) = f(2) = f(3) = 1$, and $f(d) - f(d-1) = \text{chi}(S_d) - \text{chi}(S_{d-1}) =$
$(d-2)(d-3)/2$. Since these relations determine the values of the formula for all d, all these formulas must be equal. Thus $f(d) = d(d^2-6d+11)/6 = (1/12)(e(S_d) + K^2) = \text{chi}(O_{S_d})$, for all smooth surfaces S_d of degree d in P^3. Thus we not only have Noether's formula for chi(O), but we also have an explicit polynomial for $\text{chi}(O_{S_d})$ as a function of d.

Topological implications of Hodge theory

We deduced Noether's formula and the equation chi(O) =
$d(d^2-6d+11)/6$, assuming only elementary adjunction as proved in Shafarevich, vol 1, and some elementary topology. If we assume also Hodge/Dolbeault theory, $\text{chi}(O) = 1 - h^{1,0} + h^{2,0}$, where $h^{i,0}$ = dimension of space of holomorphic i - forms. Since adjunction gives a formula for $h^{2,0}$, comparing these formulas shows that $\text{chi}(O) = 1+h^{2,0}$, hence
$h^{1,0} = 0$ for every smooth surface S in P^3. Then again by Hodge theory, the topological cohomology group $H^1(S,C) = 0$, and by universal coefficients $H^1(S,Z) = 0$, which is one of the facts also deducible from Lefschetz theory.

Extending the argument:

We could prove Noether's formula for any smooth surface S by this method, projecting S into P^3 and computing chi, the Euler characteristic, and canonical divisor, for a non singular blowup, taking into account how these invariants change under blowup. This is harder than for curves, since a surface projected to P^3 acquires a node curve having also cusps and triple points, so the nonsingular surface differs more from its projected model than does a curve. This is carried out in Griffiths and Harris' book, and in Ragni Piene's paper.

The inductive argument for chi(D)-chi(O) on a surface

To obtain the full HRR for smooth surfaces, we need to show that chi(D)-chi(O) = (1/2)(D.[D-K]). From the exact sequence 0-->O-->O(D)-->$O(D)|_D$-->0 where D is a smooth curve on a smooth surface S, and the additivity of chi, we have chi(D)-chi(O) = chi($O(D)|_D$). Since on D, the restriction of D has degree D.D, the RRT on D gives us chi($O(D)|_D$) = 1-g + D.D, where g = g(D). Since K_D = $(K_S+D)|_D$ also implies that 1-g = -(1/2)(K_S+D).D, we get chi(D) -chi(O) = D.D - (1/2)D.(K+D) = (1/2)[D.(D-K)]. Then from Bertini's's theorem that every divisor is equivalent to a difference of smooth curves, one can deduce the theorem for a general divisor D.

Cor: HRR for a smooth surface S in P^3:
chi(D) = (1/2)[D.(D-K)] + (1/12)(K^2 + e(S)).

As we said, this formula is true for all smooth projective algebraic surfaces, and all divisors, but we have proved it, assuming the adjunction formula for hypersurfaces, only for smooth surfaces in P^3 and a divisor D equivalent to a smooth curve on S.

V. Statement of the general HRR theorem in dimension n

Computing chi(O) in terms of the Todd class.

The proofs above of Noether's formula for curves and surfaces required knowing the answer for Noether's formula in advance, i.e. the formulas chi(O) = 1-g, for curves, and chi(O) = (1/12)(e(S) + K^2) for surfaces, and then finding polynomials for these expressions. We did not say where the formulas for chi(O) came from, although we did show how to compute the polynomial expressions for it. We knew Noether's formula for curves and surfaces, but what is it in general? Hirzebruch developed a useful formalism for expressing them all.

To prove such formulas in all dimensions one has to note some kind of pattern. One characteristic of chi(O) is multiplicativity, i.e. chi($O_{X\times Y}$) = chi(O_X).chi(O_Y). If one has enough such properties to characterize the formula chi(O), one can look for a sequence of polynomials with these properties, and these must be formulas for chi(O).

Chern roots of X

The following computation turns out to work. Let X be a smooth projective variety of dimension n and let $\gamma_1,\dots,\gamma_n$ be the chern roots of X, i.e. formal symbols such that the chern classes of T_X are the elementary symmetric functions in the gammas. E.g. if T_X is a direct sum of line bundles, the gammas are the first chern classes of those line bundles. It does not matter in the following discussion if you know what chern classes mean, but it is nice to know the simplest case c_1. If M is a line bundle on X, and s is a meromorphic section of M, then the first chern class of M, $c_1(M)$, is the cohomology class Poincare dual to the divisor of s, i.e. to the divisor (s) = $(s)_{zero}$ - $(s)_{pole}$. It is also basic that a vector bundle E of rank r on X, has r+1 chern classes c_0 = 1, $c_1,...,c_r$, where c^i lies in $H^{2i}(X,Z)$. The total chern class c(E) is the sum $c(E) = 1+c_1+...+c_r$ in the graded ring $H^*(X,Z)$. The Whitney product formula for a direct sum of bundles says that c(E+F) = c(E)c(F). This is useful for calculating one of these when the other two are known. The chern class of X is the chern class of its tangent bundle c(TX).

Todd class of X

The Todd class of X is a certain polynomial in its chern classes. We will define it as a power series in the chern classes which of course terminates since the ring $H^*(X,Z)$ is zero in degrees above $2\dim_C(X)$. Since we want a polynomial with certain multiplicative properties, it is not entirely shocking that the exponential series appears in this context. We will also need to invert certain power series formally. Recall the geometric series formula $1/(1-x) = 1+x+x^2+.....$ tells how to invert any power series 1- xf(x) with constant term 1, as
$1/[1-xf(x)] = 1 + xf(x) + [xf(x)]^2 + ...$

The power series $1-e^{-x}$ has zero constant term, and linear term equal to x, so is not invertible, hence we cannot define $x/(1-e^{-x})$ to be $x(1-e^{-x})^{-1}$. But $(1-e^{-x})/x$ has constant term equal to 1, hence is invertible. So we define $x/(1-e^{-x})$ = $[(1 - e^{-x})/x\,]^{-1}$ = Q(x). Then let $\tau(x) = [Q(\gamma_1).....Q(\gamma_n)]$ = the "Todd class" of X, where the gammas are the chern roots of X. This expression is symmetric in the gammas, hence can be expressed as a function of the chern classes. This lives in the cohomology ring $H^*(X,Q)$, where Q is the rational numbers, since we needed to use denominators to invert the power series.

General Noether formula

Hirzebruch proved the formula of Todd (who assumed an unproved lemma of Severi), that chi(O) equals the homogeneous part of degree n of the Todd class, i.e. for a smooth projective variety of dimension n,
$\mathbf{chi(O_X) = [\tau(x)\,]_n}$ = $[Q(\gamma_1).....Q(\gamma_n)]_n$. This is a symmetric function in the chern roots, hence expressible in terms of chern classes. This tedious algebraic task is made easier by formulas in chapter 1 of Hirzebruch.

HRR in dimension n: If D is a divisor with chern class d, on the smooth projective n dimensional variety X, i.e. $c_1(O(D))$ = d, then
$\mathbf{chi(O(D)) = [e^d.\,\tau(x)]_n}$ = the homogeneous part of degree n of the product $[e^d.\tau(x)]$, in the cohomology ring $H^*(X,Q)$, (evaluated on the fundamental class of X to give a rational number, which by this theorem is always an integer).

These power series can be calculated by hand for low dimensions:
arithmetic genera:
chi(O) = $(1/2)c_1$ = 1-g for a curve,
chi(O) = $(1/12)(c_1^2+c_2) = (1/12)(K^2+e)$ for a surface,

chi(O) = (1/24)(c_1c_2) for a threefold.

euler characteristics of general divisors:
chi(D) = d+(1/2)c_1 = d+1-g, for curves,

chi(D) = (1/2)(d^2 + dc_1) + (1/12)(c_1^2+c_2)

= (1/2)(D.[D - K]) + (1/12)(K^2 +e), for surfaces, and

chi(D) = (1/12)(2d^3 + 3d^2c_1 + d[c_1^2+c_2]) + (1/24)c_1c_2, for threefolds.

Chern classes of smooth projective hypersurfaces

These invariants are computable for a smooth hypersurface X_n in projective space by the Whitney product formula. We define the total chern class c(E) of a rank k bundle E in the graded ring $H^*(X)$ as the sum 1+c_1+...+c_k, and then c(E+F) = c(E)c(F). If X is a hypersurface in P^r, then T(P^r) = T(X)+N, where T denotes tangent bundle, and N is the normal bundle of X in P^r. Since one can compute the total chern class of T(P^r) is $(1+h)^r$ where h = c_1(O(1)) is the class in $H^2(P^r,Z)$ dual to a hyperplane, and the normal bundle N of X_n is O(n)$|_X$, we can solve for c(T_X). Thus c(X) = (1+c_1+...+c_{n-1}) = c(P^r).$(c(O(n)))^{-1}$ = $(1+h)^r.(1+nh)^{-1}$, where $(1+nh)^{-1}$ = 1-nh +n^2h^2-n^3h^3+ -.....

Invariants of smooth threefolds in P^4

For a smooth threefold X_n of degree n in P^4,

c_1 = (5-n)h,

c_2 = (10-5n+n^2)h^2,

c_3 = (10-10n+5n^2-n^3)h^3 = e(X), and

chi(O_X) = (1/24)c_1c_2 = (1/24)n(5-n)(10-5n+n^2),

since h^3 acts on X as intersection with a line, hence has value n.

Intermediate Jacobians of Cubic threefolds in P^4

It follows from arguments below that chi(O) is a birational invariant over C, but chi(O) = 1 for a cubic threefold X_3, so the arithmetic genus does not distinguish X_3 birationally from P^3. It was long believed that X_3 was not birational to P^3, but a correct proof eluded everyone for years. Then it was noticed that since P^3 has no 3rd cohomology, if X_3 were birational to P^3 it would place a subtle geometric restriction on $H^3(X_3)$ as follows. Hironaka observed that if X_3 were birational to P^3 blownup say along a curve C, then the structure of $H^3(X)$ would parallel that of $H^1(C)$. Since the topological euler characteristic of a cubic threefold is $c_3(X_3)$ = -6, and the other betti numbers agree with those for P^4, we get $b_0 = b_2 = b_4 = b_6 = 1$, and $b_1 = b_5$ = 0. Thus b_3 = 6+4 = 10, and since $b_3 = h^{0,3} + h^{1,2} + h^{2,1} + h^{3,0}$ by Hodge theory where $h^{0,3}$ = $h^{3,0}$ = dim(K) = dimH^0(O(-2)) = 0, we see that $h^{2,1}$ = 5. Then $H_3(X_3,Z)$ is a lattice with a symplectic intersection pairing, in the 5 dimensional complex vector space $H^{2,1}(X)^*$.

Non rationality of a smooth cubic threefold

By Hironaka's argument, if X_3 were birational to P^3, there would exist a genus 5 curve C, and a complex isomorphism of $H^{2,1}(X)^*$ with $H^0(C,K)^*$sending the lattice $H_3(X_3,Z)$ to the lattice $H_1(C,Z)$, and preserving the intersection pairings. Griffiths and Clemens showed no such isomorphism is possible as follows. Taking the quotient of the complex vector space by the lattice gives a compact complex torus, (the Jacobian of C, or intermediate Jacobian of X), and the symplectic pairing defines a cohomology class on the torus which determines up to translation a unique "theta" divisor Θ on the torus. An isomorphism as above preserving the symplectic pairings on lattices would induce an isomorphism of tori preserving the theta divisors, up to translation, hence preserving the Gauss maps of the theta divisors.

C-G computed the Gauss map on the theta divisor for the threefold X_3 and showed it had a different branch locus from that computed by Andreotti in the case of the theta divisor for a curve. Amazingly, the branch locus of the Gauss map for $\Theta(X)$ was the "dual variety" of tangent hyperplanes to X, just as the branch locus of the Gauss map for $\Theta(C)$ was the dual variety for the canonical model of C. In particular, since a complex projective variety is determined by its dual variety, the varieties X and C are completely determined by their Jacobians, i.e. by the periods of integrals of their middle dimensional differential forms, so Torelli's theorem holds for cubic threefolds as well as curves.

This illustrates the fact of life that, useful as they are, integer valued invariants like chi(O) usually suffice to distinguish only radically different varieties. Varieties which are very similar, such as hypersurfaces of degree $\leq n+1$ in P^n, may require extremely subtle measures to distinguish them. According to Kolla'r, it is thought that probably no smooth projective hypersurface of degree ≥ 4 is ever birational to a projective space, but there is apparently very little evidence either way.

Birational invariance of the arithmetic genus in characteristic 0

We have seen that chi(O) is a linear deformation invariant, i.e. constant for hypersurfaces in the same linear series on some ambient variety, and (by Noether's formula) even a diffeomorphism invariant. One can show over the complex numbers, and more generally in characteristic zero, chi(O) is also a birational invariant. The analytic argument is as follows.

chi(O) = $h^0(O) - h^1(O) + h^2(O) - + \ldots\ldots\ldots h^n(O)$, which by Dolbeault theory = $h^{0,0} - h^{0,1} + h^{0,2} - + \ldots\ldots h^{0,n}$, which by Hodge duality

$= h^{0,0} - h^{1,0} + h^{2,0} - + \ldots\ldots h^{n,0}$, which by Dolbeault theory

$= h^0(O) - h^0(\Omega^1) + h^0(\Omega^2) - + \ldots\ldots h^0(\Omega^n)$.

The dimensions $h^0(\Omega^i)$ of the spaces of i-forms, for $0 \leq i \leq n$, are birational invariants even in positive characteristic (using the Hartogs principle, as in Shafarevich BAG), but we have Hodge theory to equate them with the numbers $h^i(O)$ only over the complex numbers. For smooth complex projective varieties the $h^i(O)$ are birational invariants, as is chi(O). I.e. $h^i(O) = h^{0,i} = h^{i,0} = h^0(\Omega^i)$ for all i, $0 \leq i \leq \dim(X)$.

Using birational invariance, we could compute chi(P^2) in section IV above without Cech cohomology, since P^2 is birational to a smooth quadric surface Q in P^3. Since Q is linearly equivalent in P^3 to the transverse union of two copies of P^2, and since chi(P^1) = 1, it follows from chi(P^2) = chi(Q) = chi(P^2)+chi(P^2)- chi(P^1), that chi(P^2) = chi(P^1) = 1. This is the analog of the argument we gave for the plane curve P^1. The same argument shows that chi(P^n) = 1 for all n, since every smooth quadric is birational to projective space.

Exercises using RRT for curves
Assume these facts:

If D is a divisor on a curve C and L(D) the space of meromorphic functions f with div(f)+D ≥ 0, then the projective space P(L(D)) is isomorphic to the space |D| of effective divisors of form E = D+div(f) for f in L(D), by the map taking [f] to div(f)+D.

The map f:C-->$|D|^*$ taking a point p to the hyperplane of divisor E with p contained in E, is everywhere defined on C if and only if no point of C is contained in all divisors of |D|, if and only if for all p, dimL(D-p) < dimL(D). In this case the image curve spans $|D|^*$ and degf(C).deg(f) = deg(D).

If well defined, the map C-->|D| is injective if for all p ≠ q there is a divisor in |D| containing p but not q, iff dimL(D-p-q) < dimL(D-p). A well defined map is an embedding if it is injective and for all p, some divisor contains p but not 2p, i.e. if and only if for all p, dimL(D-2p) < dimL(D-p).

Prove:

(i) The map f:C-->$|D|^*$ is an embedding iff for all points p,q of C, dimL(D-p-q) = dimL(D)-2.

(ii) C is isomorphic to P^1 iff there is a one point divisor p on C such that dimL(p) = 2.

(iii) For g ≥ 1, the map associated to a canonical divisor K is always well defined, and fails to be an embedding iff there exists a divisor of degree 2 on C, D = p+q, such that dimL(p+q) = 2.

(iv) For g ≥ 2, if there is a divisor p+q with dimL(p+q) = 2, then any other such divisor belongs to |p+q|. (Hint: the map defined by K factors through the map defined by |p+q| and by the Veronese map P^1-->P^{g-1}.)

(v) If C is embedded as a spanning curve of degree 2g-2 in P^{g-1} then any hyperplane cuts on C a canonical divisor.

(vi) If C is a smooth plane quartic curve, then C has no divisor p+q with dimL(p+q) = 2.

(vii) If C has genus 1, then any divisor D of degree 3 defines an embedding of C as a smooth plane cubic.

(viii) If C is a smooth curve of genus 4 and degree 6 spanning P^3, show C lies on a quadric surface Q, and on a cubic surface F not containing Q.

(ix) Show any divisor of degree ≥ 2g+1 on C gives an embedding.

(x) If there exists a 3 to 1 branched cover f:C-->P^1, but no such 2 to 1 cover, then prove the 3 points of every fiber f^{-1}(y) over a point y of P^1, lie on a trisecant of the "canonical model" of C defined in (iii).

An exercise using RRT for a surface. Let S be a smooth cubic surface in P^3, let m be a line on it, and let C be a conic such that m+C is a plane section of S. Assume that (- m - C) = K is a canonical divisor on S. Considering any other plane section through m, cutting m+D, where D is another conic, argue that C.D = 0 = C.C. Using the fact that $chi_{top}(S) = 9$, compute $chi(O_S(C))$ = 2. Try to check the hypothesis of the vanishing theorem for C to show in fact chi(C) = $h^0(C)$ = dimL(C). This proves there is a map from S to P^1 having C as a fiber. We have seen such a map above, namely projection from m. This also proves more, i.e. that this projection map uses all the functions in L(C). Thus any meromorphic function on S with zero divisor equal to one of the conics incident to m must have as pole divisor some other one of those conics. I.e. the pencil of conics occurring as fibers of this map is a maximal linear family of curves on S.