Topology Qualifying Exam
Spring 2022

Instructions

1. Complete **8** out of **9** problems and clearly mark which problem you do not want us to grade.
2. You may assume the fundamntal group and homology groups of a *point*, the *2-torus*, and *spheres* of all dimensions. Everything else should be computed.

Questions

1. Show that if X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.
2. Let $g : X \to Y$ be a map with Y compact and Hausdorff. Show that the graph
$$\Gamma_f := \{(x, f(x)) : x \in X\} \subset X \times Y$$
is closed if and only if f is continuous.
3. Give an example where X is a Hausdorff and Y is a quotient space of X but Y is not Hausdorff.
4. Let $X = S^1 \vee S^1$ be the wedge of two circles:
   (a) Compute $\pi_1(X)$
   (b) Exhibit three distinct 2-fold covers of X up to equivalence
   (c) Show that fundamental group of *every* 2-fold cover is isomorphic to the free group on 3 letters.
5. Find the universal cover of (a) the torus, and (b) the Klein bottle. In both cases, describe how the group of deck transformations acts.
6. Compute the homology of the 'sausage link'. Specifically, let
$$2B = \left(S^2 \coprod S^2\right) / \{n \sim n' \text{ and } s \sim s'\}$$
where n, n' are the north poles on the two 2-spheres and s, s' are the two south poles.
7. Let T_1, T_2 denote two copies of the solid torus $S^1 \times D^2$, with coordinates $\{\psi\} \times \{r, \theta\}$. For p, q relatively prime, the lens space $L(p, q)$ is union of T_1 and T_2 along their boundary by the gluing map $\phi : \partial T_1 \to \partial T_2$ defined as
$$\phi(\psi, 1, \theta) = (q\psi + p\theta, 1, b\psi + a\theta)$$
for $aq - bp = 1$.
   (a) Use the Van Kampen theorem to compute the fundamental group of $L(p, q)$ in terms of p and q
   (b) Use the Mayer-Vietoris sequence to compute the singular homology of $L(p, q)$ in terms of p and q.
8. Show that $S^2 \vee RP^3$ and $S^3 \vee RP^2$ have the same fundamental group, but are not homotopy equivalent.
9. Use the Lefschetz fixed point theorem to prove that a map $f : S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map.