Algebra Qualifying Exam, Fall 2008

1. Let R be a commutative ring and M an R-module, $M \neq \{0\}$.
i) Say what it means for M to have a *basis*.
ii) Prove that if R is a field, then M has a (not necessarily finite) basis. Indicate where the hypothesis that R is a field is used.

2. Prove Gauss's lemma: The product of primitive polynomials in $\mathbb{Z}[x]$ is primitive. (A polynomial is said to be primitive if the greatest common divisor of its coefficients is 1.)

3. Let H and K be subgroups of a group G, such that K is normal in G.
i) Prove that HK is a subgroup of G.
ii) Prove that $HK/K \simeq H/(H \cap K)$.

4. Let E/F be a Galois field extension, and let K/F be an intermediate field of E/F. Prove that K is normal over F if and only if $Gal(E/K)$ is a normal subgroup of $Gal(E/F)$.

5. Let A be an $n \times n$ matrix over $\mathbb{C}$, such that $A^*A = AA^*$. Prove that A is diagonalizable.

6. Classify all groups of order 55.

7. Let $M = \mathbb{R}[x]/(x-2)(x+1) \oplus \mathbb{R}[x]/(x-2)(x^2+3)$. Let $T : M \to M$ denote the $\mathbb{R}$-linear transformation "multiplication by x." Find the following for T: i) minimal polynomial ii) characteristic polynomial iii) determinant iv) rational canonical form.

8. Let $\zeta_{11} = e^{\frac{2\pi i}{11}}$ (so ζ_{11} is a primitive 11th root of unity).
i) Prove that $\mathbb{Q}(\zeta_{11})$ is a Galois extension of $\mathbb{Q}$ and describe the Galois group of this extension.
ii) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{11})$ and write each in the form $\mathbb{Q}(\alpha)$ for some α. Prove your answers.