Topology Qualifying Exam - Spring 2012

Thursday, January 5th, 1:30-4:30

The following nine questions have equal weight (11 points each) - answer all of them.

Q1: For any integer $n \geq 2$ let X_n denote the space formed by attaching a 2-cell to the circle S^1 via the attaching map $a_n : S^1 \to S^1$ defined by $e^{i\theta} = e^{in\theta}$. (In other words, X_n is the quotient of the disjoint union of the disk D^2 and the circle S^1 by the relation which identifies a point $e^{i\theta}$ on the boundary of D^2 with $e^{in\theta} \in S^1$.)

(i): Compute the fundamental group and the homology of X_n. (6 points)

(ii): Exactly one of the X_n (for $n \geq 2$) is homeomorphic to a surface. Identify, with proof, both this value of n and the surface that X_n is homeomorphic to (including a description of the homeomorphism). (5 points)

Q2: **(i):** State what it means for a covering space to be *normal* (sometimes this condition is instead called *regular*). (4 points)

(ii): Let Θ be the topological space formed as the union of a circle and a diameter of the circle (so this space looks exactly like the letter Θ). Give an example of a covering space of Θ which is *not* normal. (7 points)

Q3: Give a self-contained proof that the 0th singular homology $H_0(X)$ is isomorphic to $\mathbb{Z}$ for every path-connected space X.

Q4: **(i):** Prove that for every continuous map $f : S^2 \to S^2$ there is some x such that either $f(x) = x$ or $f(x) = -x$. (*Hint*: Where $A : S^2 \to S^2$ is the antipodal map, you are being asked to prove that either f or $A \circ f$ has a fixed point.) (7 points)

(ii): Exhibit a continuous map $f : S^3 \to S^3$ such that for every $x \in S^3$, $f(x)$ is equal to neither x nor $-x$. (*Hint*: It might help to first think about how you could do this for a map from S^1 to S^1.) (4 points)

Q5: Suppose that U and V are open subsets of a space X, with $X = U \cup V$. Find, with proof, a general formula relating the Euler characteristics of X, U, V, and $U \cap V$.

Q6: Prove that any finite *tree* is contractible, where a tree is a connected graph that contains no closed edge paths.

Q7: Let X be a topological space.

(i): State what it means for X to be *Hausdorff*. (3 points)

(ii): Prove that every compact subset of a Hausdorff space is closed. (8 points)

Q8: Let X be a topological space.

(i): State what it means for X to be *compact* (3 points).

(ii): Let

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$

Is X compact? (4 points)

(iii): Let

$$X = (0, 1].$$

Is X compact? (4 points)

Q9: Let X be a path connected topological space.

(i): Prove that X is connected. (5 points)

(ii): Is the converse true? Prove or give a counter example. (6 points)