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Ph.D. Prelim PROBABILITY THEORY MAY 9, 1990

1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F}$ an algebra on Ω such that $\mathcal{A} = \sigma(\mathcal{F})$. Prove that, for every $\epsilon > 0$ and $A \in \mathcal{A}$, there is a set $F \in \mathcal{F}$ such that $P(A \,\Delta\, F) < \epsilon$.

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{P}$ be a π-system, (i.e., $\mathcal{P}$ is a system of events closed under finite intersection), such that $\Omega = \bigcup_{k=1}^{\infty} A_k$ for some sequence $\{A_k\} \subset \mathcal{P}$. Suppose that the equation $P(A \cap B \cap C) = P(A)P(B)P(C)$ holds for fixed events B and C and for all $A \in \mathcal{P}$. Show that this equation holds for all $A \in \sigma(\mathcal{P})$.

3. a) Let $X, Y, \in L^1(\Omega, \mathcal{A}, P)$. Show that

$$E(Y) - E(X) = \int_{-\infty}^{\infty} [P\{X < t \leq Y\} - P\{Y < t \leq X\}]\, dt$$

b) Let $(X,Y]$ be a nondegenerate random interval. Show that its expected length is the integral w.r.t. t of the probability that it covers t.

4. Let $\{X_n\}$ be a sequence of independent random variables.

a) If $E(X_n) = 0,\ n \geq 1$, and $\sum_1^{\infty} \mathrm{var}(X_n) < \infty$, show that $\sum_1^{\infty} X_n$ converges a.s.

b) Show that $S_n = \sum_{k=1}^{n} X_k$ converges a.s iff it converges in probability.

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5. Find uniformly integrable random variables X_n for which there is no integrable random variable Z satisfying $P[|X_n| \geq a] \leq P[|Z| \geq a]$, for $a > 0$. [Hint: Let X_n take values n and 0 with probabilities $p_n = \frac{1}{n\log n}$ and $1 - p_n$, respectively.]

6. a) Quote Lindeberg condition and Lyapunov condition for the central limit property of a sequence of random variables.

   b) Let $X_1, X_2, \ldots,$ be a sequence of independent normally distributed random variables with $E(X_k) = 0, k \geq 1$, and $\text{var}(X_1) = 1, \text{var}(X_k) = 2^{k-2}, k \geq 2$. Show that $\{X_k\}$ does not satisfy the Lindeberg condition.

   c) Give a sequence of random variables satsifying the Lindeberg condition but not the Lyapunov condition.

7. Use strong Law of large numbers to show that every number in [0, 1) is normal (i.e., with probability 1 the proportion of zeros and ones in the binary expansion of the numbers tends to $\frac{1}{2}$).

8. Let $\{\mathcal{F}_n\}$ be a nonincreasing family of σ-algebras, $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots$, and X be an integrable random variable. Show that

$$E\{X|\mathcal{F}_n] \longrightarrow E[X|\mathcal{F}_\infty] \text{ P-a·s and in } L^1,$$

where $\mathcal{F}_\infty = \bigcap_{k=1}^{\infty} \mathcal{F}_k$.