PRELIMINARY EXAMINATION IN TOPOLOGY

Fall 1998

Directions: Do all of problems 1-6 and two of problems 7-9. Each problem is worth 10 points, except for problems 5 (25 points) and 7 (15 points).

A. Do all of the next three problems.

1. (10 points) Let $C \subset [0,1]$ be the Cantor set. Prove that C is a retract of no open subset of $[0,1]$.

2. (10 points) Let $f: X \to Y$ be a surjective local homeomorphism.
   a. Prove that if X is compact and Y is Hausdorff, then f is a covering map.
   b. Give examples to show both hypotheses in a. are necessary.

3. (10 points) Prove or give a counterexample:
   a. A separable metric space is second countable.
   b. A separable first countable space is second countable.

(Recall that a space is *second countable* if there is a countable basis for its topology; it is *first countable* if there is a countable basis at each point.)

B. Do all of the next three problems.

4. (25 points) Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$.
   a. Calculate $\pi_1(X)$, showing your work, and describe the universal covering space of X.
   b. Exhibit a two-sheeted covering space Y of X with $\pi_1(Y) \cong \mathbb{Z}$.
   c. Calculate $H_*(X, \mathbb{Z})$.

5. (10 points) State and *sketch* a proof of the path lifting property of covering spaces.

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6. (15 points) Define a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be endowed with the quotient topology.

a. Prove that A induces a well-defined, continuous map $\overline{A}: T^2 \to T^2$.

b. Suppose $f: T^2 \to T^2$ is a continuous map that is homotopic to $\overline{A}$. Must f have a fixed point? (Proof?)

C. Do <u>two</u> of the next three problems.

7. (10 points) Let $n \geq 2$. Can there be a continuous function $f: S^n \to S^1$ with the property that $f(-x) = -f(x)$ for all $x \in S^n$? Proof?

8. (10 points) Let $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Suppose $f: D^n \to \mathbb{R}^n$ is continuous and $\|f(x) - x\| < 1$ for all $x \in \partial D^n$. Prove that there is a point $x \in D^n$ with $f(x) = 0$.

9. (10 points) Use covering space techniques to prove that if G is a free group on n generators and $H \subset G$ is a subgroup of index d, then H is a free group. Give a formula for the number of generators of H in terms of n and d.