① Cauchy-Schwarz inequality $\Rightarrow$ for $0 \le h \le t$

$$\left(E|X|^{t}\right)^{2} \le E\left(|X|^{t+h}\right) \cdot E\left(|X|^{t-h}\right)$$

$$\overset{\shortparallel}{E}\left(|X|^{\frac{t+h}{2}} \cdot |X|^{\frac{t-h}{2}}\right)$$

Setting $t_1 = t+h$, $t_2 = t-h$

$$\Rightarrow \left(E|X|^{\frac{t_1+t_2}{2}}\right)^{2} \le E\left(|X|^{t_1}\right) E\left(|X|^{t_2}\right)$$

$$\Rightarrow 2\log E|X|^{\frac{t_1+t_2}{2}} \le \log E|X|^{t_1} + \log E|X|^{t_2}$$

$$\Rightarrow g\left(\frac{t_1+t_2}{2}\right) \le \frac{g(t_1)+g(t_2)}{2} \Rightarrow g(t) \text{ convex } (g(t) \text{ continuous})$$

Note that $g(0)=0$.

$$\frac{g(t)}{t} = \frac{g(t)-g(0)}{t-0} \uparrow \text{ in } t \Rightarrow \left(E|X|^{t}\right)^{\frac{1}{t}} \uparrow \text{ in } t$$

② If X, Y independent, then

$$E f(X) g(Y) = E f(X)\, E g(Y)$$

for simple functions f, g

$$\left(f = \sum a_i I_{A_i}, \quad g = \sum b_j I_{B_j} \right)$$

approximate continuous f, g by simple functions.

If $E f(X) g(Y) = E f(X)\, E g(Y)$, then choose

continuous f, g to approximate I_A & I_B

③ It suffices to show $P\left(\left|\frac{S_n - np}{n^\beta}\right| > \frac{1}{k} \text{ i.o.}\right) = 0, \quad \forall k = 1, 2, \cdots$

Let $S_n^* = \frac{S_n - np}{\sqrt{npq}}$, then

$$P\left(\left|\frac{S_n - np}{n^\beta}\right| > \frac{1}{k} \text{ i.o.}\right) = P\left(|S_n^*| > \frac{n^{\beta - \frac{1}{2}}}{k\sqrt{pq}} \text{ i.o.}\right)$$

$$\leq P\left(|S_n^*| > 2\sqrt{\log n} \text{ i.o.}\right)$$

Sufficient $\Rightarrow$ $\sum P(|S_n^*| > 2\sqrt{\log n}) < \infty$

Use normal approximation to obtain

$$P(|S_n^*| > 2\sqrt{\log n}) = P(S_n^* > 2\sqrt{\log n}) + P(S_n^* < -2\sqrt{\log n})$$

$$\sim \frac{1}{2\sqrt{\log n}} \varphi(2\sqrt{\log n}) + \frac{1}{2\sqrt{\log n}} \varphi(2\sqrt{\log n})$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$\Rightarrow \sum P(|S_n^*| > 2\sqrt{\log n}) < \infty$$

④ Take $a_n = \frac{(-1)^n}{n}$, Then $a_n = O(\frac{1}{n})$, $\sum a_n$ converges.

Assume $X_1 \geq 0$ ~~otherwise~~ (consider X_1^+, X_1^- separately)

Set $X_n' = X_n I_{\{X_n \leq n\}}$,

$Y_n = X_n' - E X_n'$

Borel-Cantelli thm $\Rightarrow$ X_n', X_n equivalent

$E X_n' \uparrow E X_n$ $\Rightarrow$ $\sum a_n E X_n'$ converges.

Suffices to show $\sum a_n Y_n$ converges.

Set $A_j = \{ j-1 < X_1 \leq j \}$.

$$\Rightarrow \sum_n a_n^2 E Y_n^2 \leq \sum_n a_n^2 E(X')^2 \leq \sum_n \frac{1}{n^2} \int_{X_1 \leq n} X_1^2$$

$$\leq \sum_n \sum_j^n \frac{1}{n^2} \int_{A_j} X_1^2 \leq 2 E X_1$$

Khintchine-Kolmogorov Conv. thm $\Rightarrow$ $\sum a_n Y_n$ converges

(5) In view of the mean convergence criterion, it suffices to show

$$\sup_n E|X_n|^p < \infty \quad \text{provided } X_n \xrightarrow{P} X.$$

pf ① $\exists\, \delta_0 > 0$ s.t. $P(A) < \delta_0 \Rightarrow \sup_n \int_A |X_n|^p dP < 1$

② $\exists\, a > 0$, $\exists N_0$ s.t. $P(|X_n| > a) < \delta_0$, $\forall n \geq N_0$

To see ②, note that $\exists\, a_1$, $P(|X| > a_1) < \frac{\delta_0}{2}$ (b/c $P(|X| < \infty) = 1$)

( )

$\exists N_0$ s.t. $P(|X_n| > 2a_1) \leq P(|X_n - X| \geq a_1) + P(|X| > a_1) < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0$

$\forall n \geq N_0$

For $n \in N_0$

$$E|X_n|^p = E|X_n|^p I_{|X_n| \geq a} + E|X_n|^p I_{\{|X_n| \leq a\}}$$

$$\leq (1 + a^p)$$

$$\Rightarrow \sup_n E|X_n|^p < \infty.$$

⑥ choose n_m = smallest integer $\geq m^{\frac{2}{\alpha}}$ $(m = 1, 2, \dots)$

Then $Y_m := \frac{1}{n_m}(X_1 + \cdots + X_{n_m}) \xrightarrow{a.s.} 0$ as $m \to \infty$

In fact $E(Y_m)^2 \leq \frac{c}{n_m^{\alpha}} \leq \frac{c}{m^2}$ (for some c)

$$P(\max_{m \geq N} |Y_m| > \varepsilon) = P\left(\bigcup_{N}^{\infty} |Y_m| > \varepsilon\right)$$

$$\leq P(|Y_N| > \varepsilon) + \cdots$$

$$\leq \frac{1}{\varepsilon^2}\left(E(Y_N)^2 + \cdots\right) \to 0 \text{ as } N \to \infty$$

Set $Z_m = \max_{1 \leq k \leq n_{m+1} - n_m} \left| \frac{X_{n_m+1} + \cdots + X_{n_m+k}}{n_m + k} \right|$

Then $\left| \frac{X_1 + \cdots + X_n}{n} \right| \leq Y_m + Z_m$ provided $n_m \leq n \leq n_{m+1}$

Now $E Z_m^2 \leq E\left(\frac{n_{m+1} - n_m}{n_m^2} \sum_{1}^{n_{m+1} - n_m} X_{n_m + j}^2 \right)$

$$\leq \left(\frac{n_{m+1} - n_m}{n_m} \right)^2 \leq \left(\frac{(m+1)^{\frac{2}{\alpha}} + 1 - m^{\frac{3}{\alpha}}}{m^{\frac{2}{\alpha}}} \right)^2 \sim \frac{c}{m^2} \quad (\text{if } 0 < \alpha < 2)$$

(If $\alpha > 2$, take $a < \beta < 2$, $\alpha = \sigma + \beta$ $(\sigma > 0)$

then $E\left[\frac{1}{n}(X_1 + \cdots + X_n) \right]^2 \leq O\left(\frac{1}{n^{\beta}} \right)$.)

$\Rightarrow E[Z_m]^2 \leq \frac{c}{m^2} \Rightarrow Z_m \xrightarrow{a.s.} 0$

$\Rightarrow \frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} 0$

7 (a) $E[M_t - M_s | F_s]$ (for $t \ge s$), ($F_t = F_t^B$)

$= E[B_t^2 - B_s^2 - t + s | F_s]$

$= E[(B_t - B_s)^2 + 2B_s(B_t - B_s) - t + s | F_s]$

$= E[(B_t - B_s)^2 | F_s] + 2B_s E[B_t - B_s | F_s] - t + s$

$= E(B_t - B_s)^2 + 2B_s E(B_t - B_s) - t + s$

$= t - s + 0 - t + s = 0$ $\Rightarrow$ M_t is a martingale.

(b) $E[N_t - N_s | F_s]$

$= E[(B_t - B_s)^3 - 3(B_t - B_s)^2 B_s + 3(B_t - B_s) B_s^2 + 3(t-s) B_s + 3t(B_t - B_s) | F_s]$

$= E(B_t - B_s)^3 - 3B_s E(B_t - B_s)^2 + 3B_s^2 E(B_t - B_s) + 3(t-s)B_s + 0$

$= -3B_s(t-s)B_s + 3(t-s)B_s = 0$ $\Rightarrow$ N_t is a martingale.